Math 203 - Differential and Integral Calculus

Chapter 2: Multiple Integrals

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October 20, 2026

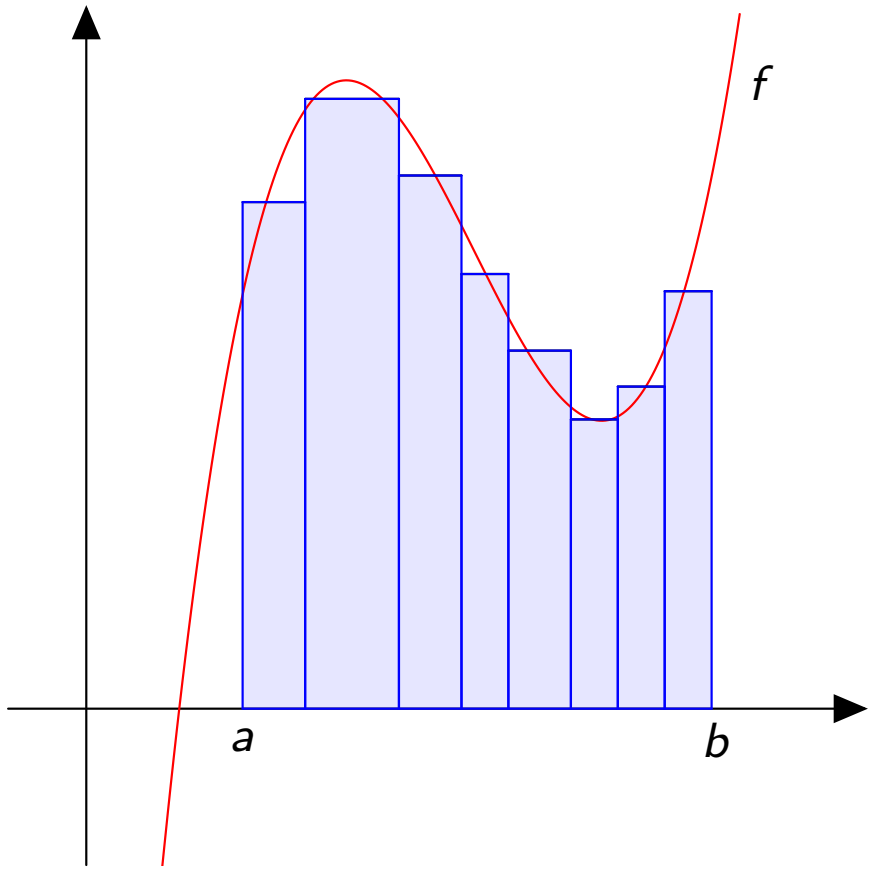
1. Double integrals
   - Areas and volumes
   - Double integrals in polar coordinates
   - Surface area
2. Triple integrals
3. Triple integrals
   - Triple integrals in Cartesian coordinates
   - Volumes
   - Triple integrals in cylindrical coordinates
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4. Physical applications of integrals
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Riemann Sum of a function of one variable

Recall that, for a function $f : [a, b] \to \mathbb{R}$, a partition $\mathcal{P} = \{a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$ of $[a, b]$ and a set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, where $\omega_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, the Riemann sum is defined by

$$\mathcal{R}(f, \mathcal{P}, \Omega) = \sum_{k=1}^{n} f(\omega_k)(x_k - x_{k-1}).$$

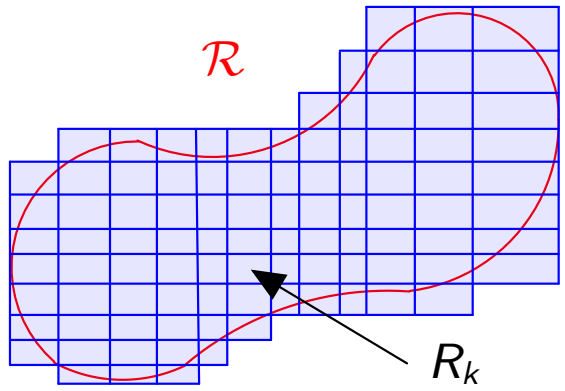
If $f(x) \geq 0$ on $[a, b]$, this is the sum of the areas of the blue rectangles. If f is continuous on $[a, b]$, then the limit



$\lim_{||\mathcal{P}|| \to 0} \mathcal{R}(f, \mathcal{P}, \Omega)$, where $||\mathcal{P}|| = \max(x_k - x_{k-1})$ exists. It is equal by definition to $\int_a^b f(x)dx$, the definite integral of f.

Riemann Sum of a function of two variables

Let $\mathcal{R}$ be a region bounded by a closed curve in $\mathbb{R}^2$. For a function $f : \mathcal{R} \to \mathbb{R}$, consider a partition of $\mathcal{R}$ into small rectangles $\mathcal{P} = \{R_1, R_2, \ldots, R_n\}$ and a set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \subset \mathcal{R}$, where $\omega_k \in R_k$ for $k = 1, \ldots, n$. Let ΔR_k be the area of the rectangles R_k, $k = 1, \ldots, n$, and $||\mathcal{P}|| = \max \Delta R_k$.



The Riemann sum is defined by $\mathcal{R}(f, \mathcal{P}, \Omega) = \sum_{k=1}^{n} f(\omega_k)\Delta R_k$. If $f(x) \geq 0$ on $\mathcal{R}$, this is the sum of the areas of cubes of bases the R_k.

If the limit $\lim_{||\mathcal{P}|| \to 0} \mathcal{R}(f, \mathcal{P}, \Omega)$ exists, we call this limit the double integral of f on $\mathcal{R}$ and denote it $\iint_{\mathcal{R}} f(x, y) d\mathcal{A}$. If f is continuous, then this limit exists.

Properties of the double integral

Let $\mathcal{R}$ be a region bounded by a closed curve in $\mathbb{R}^2$, $\mathcal{P}$ a partitions of $\mathcal{R}$ into rectangles and Ω a set of taken from each rectangle. Notice that for real functions f, f_1, f_2 defined on $\mathcal{R}$ and real number k, we have:

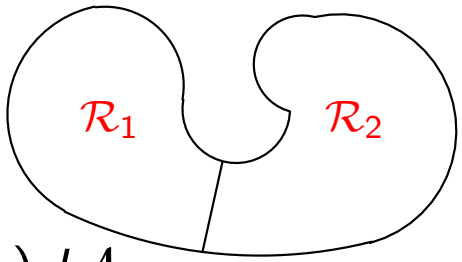
1. $\mathcal{R}(kf, \mathcal{P}, \Omega) = k\mathcal{R}(f, \mathcal{P}, \Omega)$,
2. $\mathcal{R}(f_1 + f_2, \mathcal{P}, \Omega) = \mathcal{R}(f_1, \mathcal{P}, \Omega) + \mathcal{R}(f_2, \mathcal{P}, \Omega)$,
3. If $f \geq 0$, then $\mathcal{R}(f, \mathcal{P}, \Omega) \geq 0$.

We deduce the following

1. $\displaystyle\iint_{\mathcal{R}} kf(x,y)d\mathcal{A} = k\iint_{\mathcal{R}} f(x,y)d\mathcal{A}$,
2. $\displaystyle\iint_{\mathcal{R}} (f_1(x,y) + f_2(x,y))d\mathcal{A} = \iint_{\mathcal{R}} f_1(x,y)d\mathcal{A} + \iint_{\mathcal{R}} f_2(x,y)d\mathcal{A}$,
3. If $f \geq 0$, then $\displaystyle\iint_{\mathcal{R}} f(x,y)d\mathcal{A} \geq 0$.

Properties of the double integral

4. If a curve divides a region $\mathcal{R}$ into two subregions, $\mathcal{R}_1, \mathcal{R}_2$, then



$$\iint_{\mathcal{R}} f(x,y)d\mathcal{A} = \iint_{\mathcal{R}_1} f(x,y)d\mathcal{A} + \iint_{\mathcal{R}_2} f(x,y)d\mathcal{A}.$$

Iterated double integral:
Usually, if we apply a simple integral twice, we shorten the notation by omitting the brakets.

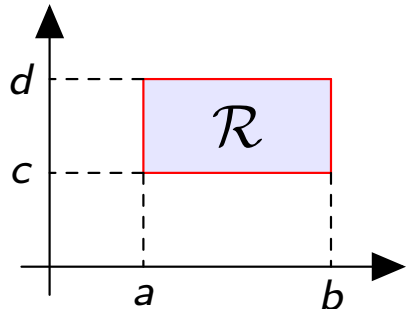
$$\text{We write } \int_a^b \int_c^d f(x,y)dxdy \quad \text{instead of} \quad \int_a^b \left(\int_c^d f(x,y)dx \right) dy.$$

This is called an iterated double integral. To compute a double integral, most of the times we use iterated double integrals.

Double integral over a rectangular region

Theorem (Fubini)

If $\mathcal{R} = [a, b] \times [c, d]$ is a rectangular region and $f : \mathcal{R} \to \mathbb{R}$ is continuous, then



$$\iint_{\mathcal{R}} f(x,y)d\mathcal{A} = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Example 1: Compute $\iint_{[0,1]\times[0,2]} \left(x^2 + xy - y^3\right) d\mathcal{A}$.

Solution: We have

$$\iint_{[0,1]\times[0,2]} \left(x^2 + xy - y^3\right) d\mathcal{A} = \int_0^2 \int_0^1 \left(x^2 + xy - y^3\right) dxdy$$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2} - xy^3\right]_0^1 dy = \int_0^2 \left(\frac{1}{3} + \frac{y}{2} - y^3\right) dy = \left[\frac{y}{3} + \frac{y^2}{4} - \frac{y^4}{4}\right]_0^2 = -\frac{7}{3}.$$

Double integral over a rectangular region

Another method: We have

$$\iint_{[0,1]\times[0,2]} \left(x^2+xy-y^3\right) d\mathcal{A} = \int_0^1 \int_0^2 \left(x^2+xy-y^3\right) dydx$$

$$= \int_0^1 \left[x^2y + \frac{xy^2}{2} - \frac{y^4}{4}\right]_0^2 dx = \int_0^1 \left(2x^2+2x-4\right)dx = \left[\frac{2x^3}{3}+x^2-4x\right]_0^1 = -\frac{7}{3}.$$

Example 2: Compute $\iint_{[0,1]\times[1,2]} \frac{1}{y^3} e^{\frac{x}{y}} d\mathcal{A}.$

Solution: We have

$$\iint_{[0,1]\times[1,2]} \frac{1}{y^3} e^{\frac{x}{y}} d\mathcal{A} = \int_1^2 \int_0^1 \frac{1}{y^3} e^{\frac{x}{y}} dxdy = \int_1^2 \left[\frac{1}{y^2} e^{\frac{x}{y}}\right]_0^1 dy$$

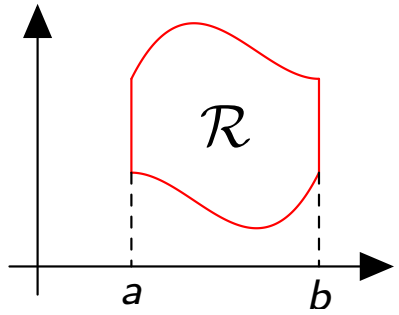
$$= \int_1^2 \left(\frac{1}{y^2} e^{\frac{1}{y}} - \frac{1}{y^2}\right) dy = \left[-e^{\frac{1}{y}} + \frac{1}{y}\right]_1^2 = e - e^{\frac{1}{2}} - \frac{1}{2}.$$

Algebraically, we don't know how to compute this double integral when starting first by integrating with respect to y.

Double integral over a non rectangular region

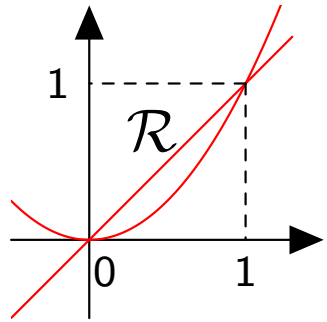
Theorem

Let $f : \mathcal{R} \to \mathbb{R}$ be a continuous function defined on the region $\mathcal{R} = \{(x, y) | a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$. We have $\iint_{\mathcal{R}} f(x, y) d\mathcal{A} = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx.$



Example 1: Compute $\iint_{\mathcal{R}} x^2 y \, d\mathcal{A}$, where $\mathcal{R}$ is the region bounded by the curves $y = x$ and $y = x^2$.

Solution: The intersection points of the two curves are the solutions of the system $\begin{cases} y = x \\ y = x^2 \end{cases}$. These are the points $(0, 0)$ and $(1, 1)$.

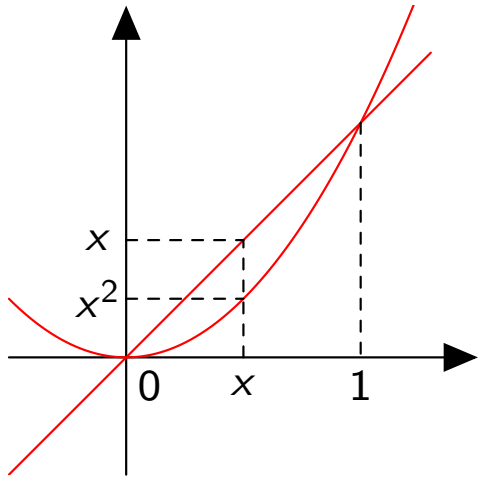


Double integral over a non rectangular region

First method: We have

$$\iint_{\mathcal{R}} x^2 y \, d\mathcal{A} = \int_0^1 \int_{x^2}^{x} x^2 y \, dy dx = \int_0^1 \left[\frac{x^2 y^2}{2} \right]_{x^2}^{x} dx$$

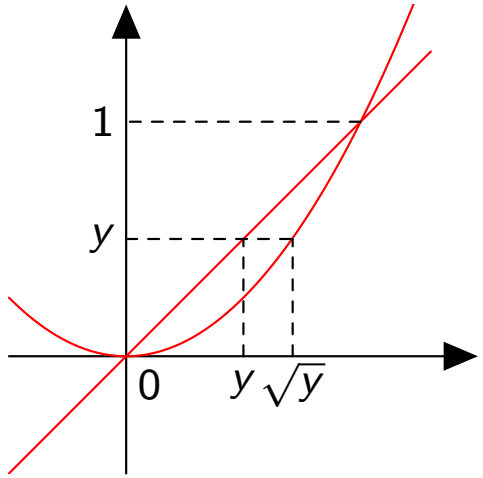
$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx = \left[\frac{x^5}{10} - \frac{x^7}{14} \right]_0^1 = \frac{1}{35}.$$



Second method: We have

$$\iint_{\mathcal{R}} x^2 y \, d\mathcal{A} = \int_0^1 \int_{y}^{\sqrt{y}} x^2 y \, dx dy = \int_0^1 \left[\frac{x^3 y}{3} \right]_{y}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y^{\frac{5}{2}}}{3} - \frac{y^4}{3} \right) dx = \left[\frac{2y^{\frac{7}{2}}}{21} - \frac{y^5}{15} \right]_0^1 = \frac{1}{35}.$$



Areas and volumes

Theorem

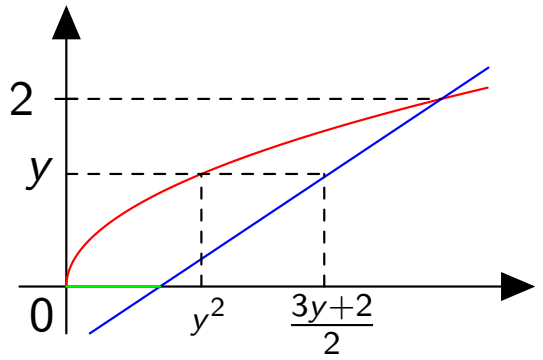
Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a bounded region and $f : \mathcal{R} \to \mathbb{R}$ a nonnegative integrable function.

1. *The area of the region $\mathcal{R}$ is given by $Area(\mathcal{R}) = \iint_{\mathcal{R}} 1 \, d\mathcal{A}$.*
2. *The volume of the solid over $\mathcal{R}$ bounded above by the surface given by f is given by the formula $Volume = \iint_{\mathcal{R}} f(x, y) \, d\mathcal{A}$.*

Example: Find the area of the region $\mathcal{R}$ bounded by the curves of equations $y = \sqrt{x}$, $y = 0$ and $2x - 3y = 2$.
Solution: We compute $\iint_{\mathcal{R}} 1 \, d\mathcal{A}$ in two ways:

1. **First way:** $\iint_{\mathcal{R}} 1 \, d\mathcal{A} = \int_0^2 \int_{y^2}^{\frac{3y+2}{2}} 1 dx dy =$

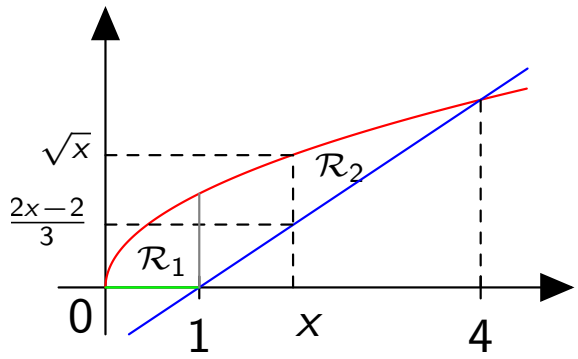
$$\int_0^2 \left(\frac{3y+2}{2} - y^2 \right) dy = \left[\frac{3y^2}{4} + y - \frac{y^3}{3} \right]_0^2 = \frac{7}{3}.$$



Areas and volumes

2. **Second way:**

$$\iint_{\mathcal{R}} 1 \, d\mathcal{A} = \iint_{\mathcal{R}_1} 1 \, d\mathcal{A} + \iint_{\mathcal{R}_2} 1 \, d\mathcal{A}$$



$$= \int_0^1 \int_0^{\sqrt{x}} 1 dy dx + \int_1^4 \int_{\frac{2x-2}{3}}^{\sqrt{x}} 1 dy dx$$

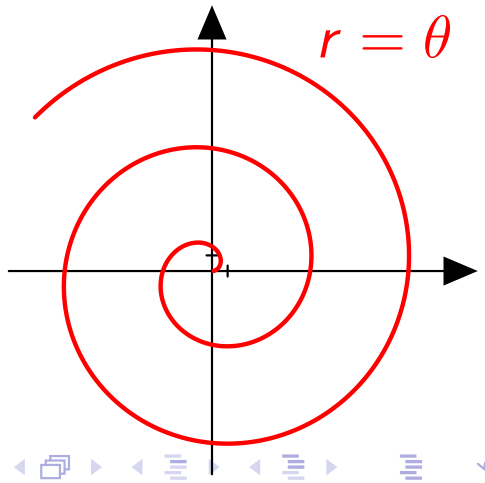
$$= \int_0^1 \sqrt{x} dx + \int_1^4 \left(\sqrt{x} - \frac{2x-2}{3} \right) dx = \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^1 + \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2 - 2x}{3} \right]_1^4$$

$$= \frac{2}{3} + \left(\frac{16}{3} - \frac{8}{3} \right) - \left(\frac{2}{3} + \frac{1}{3} \right) = \frac{7}{3}.$$

Double integrals in polar coordinates

Recall that the relation between polar coordinates (r, θ) and Cartesian coordinates is given by $x = r\cos\theta, y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$. Here are examples of equations of some curves in polar coordinates:

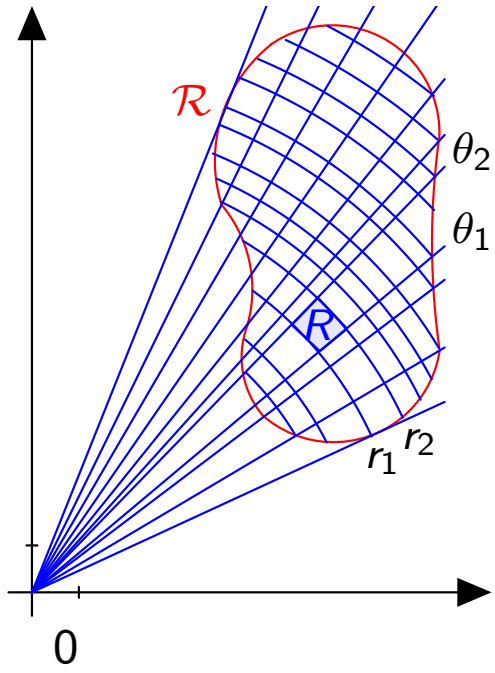
1. The parabola $y = x^2$ in polar coordinates: $r\sin\theta = r^2\cos^2\theta$ which becomes $r = \sec\theta\tan\theta$.
2. The line $x = 1$ in polar coordinates: $r\cos\theta = 1 \Leftrightarrow r = \sec\theta$.
3. The circle $x^2 + y^2 = 1$ in polar coordinates becomes $r = 1$.
4. The curve $r = \sin\theta$ becomes in Cartesian coordinates $x^2 + y^2 = y$, which is equivalent to $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$, which is a circle.
5. The equation of the spiral $r = \theta$ is written in Cartesian coordinates
   $x = \sqrt{x^2 + y^2}\cos\sqrt{x^2 + y^2}$,
   $y = \sqrt{x^2 + y^2}\sin\sqrt{x^2 + y^2}$.



Some curves have equation in polar coordinates much simpler than in Cartesian coordinates.

Double integrals in polar coordinates

In Cartesian coordinates, the region $\mathcal{R}$ is decomposed into small rectangles whose area is $\Delta x \Delta y$. This is why we have $d\mathcal{A} = dxdy$. However, in polar coordinates, $\mathcal{R}$ is decomposed into small polar rectangles $R = \{(r,\theta) | r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$. To compute the area of the polar triangle, recall that the area of a disc or radius r is πr^2, a half disc $\frac{\pi}{2} r^2$ and a sector of radius r and angle $\Delta\theta = \theta_2 - \theta_1$ is $\frac{\Delta\theta \cdot r^2}{2}$. Therefore, the area of the polar rectangle R is



$\Delta R = \frac{\Delta\theta}{2}\left(r_2^2 - r_1^2\right) = \left(r_1 + \frac{\Delta r}{2}\right)\Delta r \Delta\theta \approx r_1 \Delta r \Delta\theta$, where $\Delta r = r_2 - r_1$.
We deduce that in polar coordinates $d\mathcal{A} = rdrd\theta$.

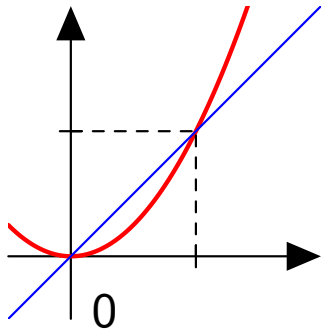
Example 1: Compute the double integral of $f : \mathcal{R} \to \mathbb{R}, (r,\theta) \mapsto r^2 \cos\theta$, where $\mathcal{R}$ is the right half side of the region bounded by $r = \sin\theta$.

Solution: We have

$$\iint_{\mathcal{R}} r^2 \cos\theta d\mathcal{A} = \int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r^3 \cos\theta dr d\theta = \int_0^{\frac{\pi}{2}} \cos\theta \frac{\sin^4\theta}{4} d\theta = \left[\frac{\sin^5\theta}{20}\right]_0^{\frac{\pi}{2}} = \frac{1}{20}.$$

Double integrals in polar coordinates

Example 2: Compute in Cartesian coordinates and in polar coordinates the area of the region $\mathcal{R}$ bouded by the parabola $y = x^2$ and the line $y = x$.



Solution: We have

1. In Cartesian coordinates:
$$\iint_{\mathcal{R}} 1 d\mathcal{A} = \int_0^1 \int_{x^2}^{x} 1 dy dx = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$
2. In polar coordinates: $\iint_{\mathcal{R}} 1 d\mathcal{A} = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta \tan\theta} r dr d\theta =$
$$\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sec\theta \tan\theta} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \sec^2\theta \tan^2\theta d\theta = \frac{1}{6} \left[\tan^3\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{6}.$$

Surface area

Let $\mathcal{R}$ be a region in $\mathbb{R}^2$ bounded by a closed curve and $f : \mathcal{R} \to \mathbb{R}$ a differentiable function with continuous partial derivatives.
The region $\mathcal{R}$ is partitioned into small rectangles $\{R_1, R_2, \ldots, R_n\}$. We choose n points $\{\omega_1, \omega_2, \ldots, \omega_n\}$ in $\mathcal{R}$, one from each small rectangle. We consider for each point ω_k, the planar parallelogram P_k over the small rectangle R_k tangent to the surface given by the graph of f at $(\omega_k, f(\omega_k))$. This tangent plane has equation $\Delta z = f_x(\omega_k)\Delta x + f_y(\omega_k)\Delta y$ and the planar parallelogram P_k has area

$$\begin{aligned}\Delta P_k &= \|(\Delta x_k, 0, f_x(\omega_k)\Delta x_k) \times (0, \Delta y_k, f_y(\omega_k)\Delta y_k)\| \\ &= \|(-f_x(\omega_k), -f_y(\omega_k), 1)\|\Delta x_k \Delta y_k \\ &= \sqrt{f_x(\omega_k)^2 + f_y(\omega_k)^2 + 1}\ \Delta R_k.\end{aligned}$$

The area of the surface defined by the graph of f over the region $\mathcal{R}$ is given by

$$A(\mathcal{S}) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta P_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{f_x(\omega_k)^2 + f_y(\omega_k)^2 + 1}\ \Delta R_k.$$

Surface area

Theorem

Let $\mathcal{R}$ be a region in $\mathbb{R}^2$ bounded by a closed curve and $f : \mathcal{R} \to \mathbb{R}$ a differentiable function with continuous partial derivatives. The area of the surface defined by the graph of f over the region $\mathcal{R}$ is given by

$$A(\mathcal{S}) = \iint_{\mathcal{R}} \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA.$$

Example 1: Find the formula for the area of a ball of radius R.

Example 2: Compute the volume of the solid defined by the graph of $f : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto \frac{e^y}{x+1}$ over the region $\mathcal{R} = \{(x,y) | 0 \leq x \leq e-1, \ln(1+x) \leq y \leq 1\}$.

Example 3: Compute the area of the surface defined by the graph of $f : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto \sqrt{x^2 - y^2}$ over the region $\mathcal{R} = \{(x,y) | 0 \leq x,\ 1 \leq y \leq 2,\ 1 \leq x^2 - y^2 \leq y^4\}$.

Triple integrals

Example 1: Evaluate the triple integral $\iiint_{\mathcal{C}} \frac{xyz - z^3}{(xy + z^2)^3} d\mathcal{V}$, where $\mathcal{C}$ is the cube $[0, 1] \times [0, 1] \times [1, \sqrt{2}]$.

Solution:

$$\iiint_{\mathcal{C}} \frac{xyz - z^3}{(xy + z^2)^3} d\mathcal{V} = \int_1^{\sqrt{2}} \int_0^1 \int_0^1 \frac{xyz - z^3}{(xy + z^2)^3} dxdydz$$

$$= \int_1^{\sqrt{2}} \int_0^1 -\frac{1}{2} \left[\frac{xz}{(xy + z^2)^2} - \frac{z^3}{y(xy + z^2)^2} \right]_0^1 dydz$$

$$= \int_1^{\sqrt{2}} \int_0^1 -\frac{1}{2} \left(\frac{z}{(y + z^2)^2} - \frac{z^3}{y(y + z^2)^2} + \frac{1}{yz} \right) dydz$$

$$= \int_1^{\sqrt{2}} \int_0^1 -\frac{1}{2} \left(\frac{2z}{(y + z^2)^2} + \frac{1}{z(y + z^2)} \right) dydz$$

$$= \int_1^{\sqrt{2}} -\frac{1}{2} \left[-\frac{2z}{y + z^2} + \frac{1}{z} \ln |y + z^2| \right]_0^1 dz$$

Triple integrals

$$= \int_1^{\sqrt{2}} -\frac{1}{2}\left(-\frac{2z}{1+z^2} + \frac{1}{z}\ln(1+z^2) + \frac{2}{z} - \frac{2}{z}\ln z\right) dz$$

Example 2: Evaluate the triple integral $\iiint_{\mathcal{T}} (x^2+y^2)d\mathcal{V}$, where $\mathcal{T}$ is the tetrahedron defined by $x, y, z \geq 0$ and $x + y + z \leq 1$.

Example 3: Evaluate the triple integral $\iiint_{\mathcal{P}} z^2 d\mathcal{V}$, where $\mathcal{P}$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Theorem

The volume of a solid $\mathcal{S}$ is given by the formula $V = \iiint_{\mathcal{S}} 1 d\mathcal{V}$.

Example 4: Compute the volume of the tetrahedron $x, y, z \geq 0$ and $x + y + z \leq 1$.

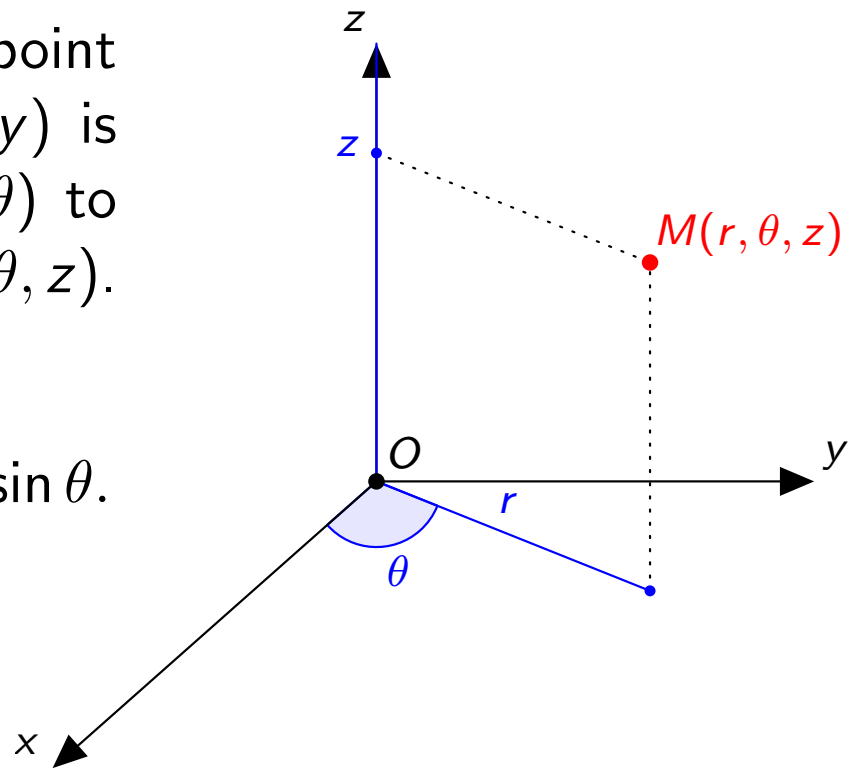
Cylindrical coordinates

Cylindrical coordinates: For a point (x, y, z) in Cartesian coordinates, (x, y) is replaced by the polar coordinates (r, θ) to obtain the cylindrical coordinates (r, θ, z). We have

$$r = \sqrt{x^2 + y^2}, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

Similarly to polar coordinates, we have

$$d\mathcal{V} = rdrd\theta dz.$$



Some surfaces in cylindrical coordinates: 1. $r = R$ a constant is a cylinder; 2. $\theta = \theta_0$ a constant is a half vertical plane; 3. $z = kr$ is a cone; 4. $z = kr^2$ is a paraboloid.

Cylindrical coordinates

Example 1: Use cylindrical coordinates to compute the volume of a cone of height h and base with radius R.

Example 2: Use cylindrical coordinates to compute the volume of solid inside the cone $z = \sqrt{x^2 + y^2}\cot\phi$ of half angle $0 < \phi < \frac{\pi}{2}$ and the sphere $x^2 + y^2 + z^2 = \rho^2$ of radius ρ.

Example 3: Evaluate the integral $\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz \; dzdxdy$ by changing to cylindrical coordinates.

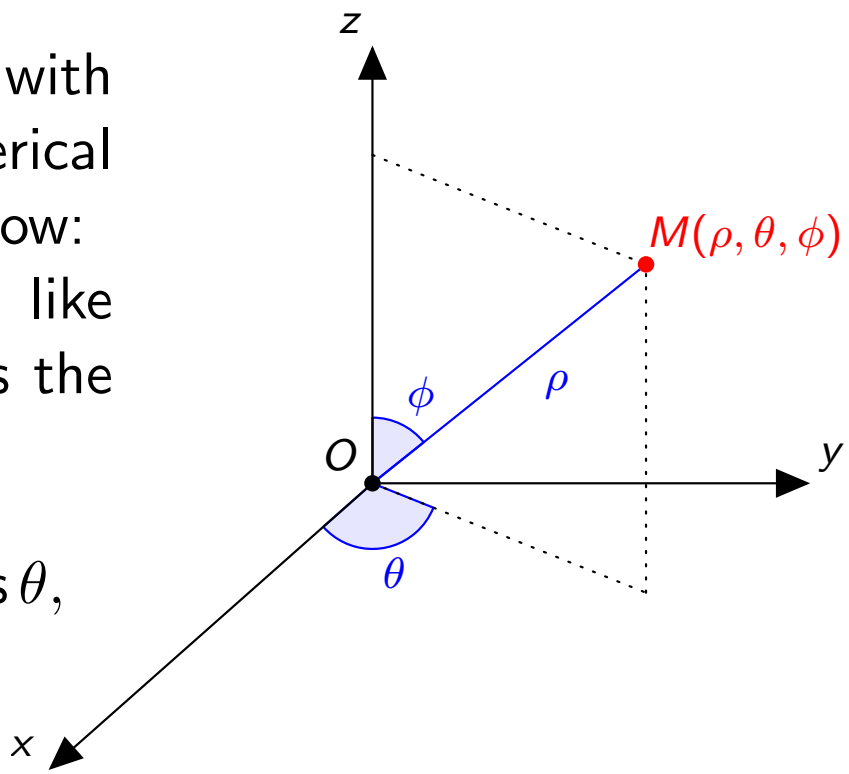
Answers: 1. $V = \frac{1}{3}\pi R^2 h$; 2. $V = \frac{2}{3}\pi\rho^3(1 - \cos\phi)$

Spherical coordinates

Spherical coordinates: For a point M with Cartesian coordinates (x, y, z), its spherical coordinates (ρ, θ, ϕ), are defined as follow: ρ is the distance from 0 to M. θ is like in cylindrical coordinates. $\phi \in [0, \pi]$ is the angle from $\vec{Oz}$ to $\vec{OM}$. We have

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad x = \rho \sin\phi \cos\theta,$$

$$y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$



Some surfaces in spherical coordinates: 1. $\rho = R$ a constant is a sphere; 2. $\phi = \phi_0$ a constant is a cone.

Spherical coordinates

We have seen that the volume of the solid inside the cone $\phi = \phi_0$ and the sphere $\rho = \rho_0$ is given by $\mathcal{V} = 2\pi\frac{\rho_0^3}{3}(1 - \cos\phi_0)$.
The volume of the solid defined by $\rho_1 \leq \rho \leq \rho_2$, $\theta_1 \leq \theta \leq \theta_2$,$\phi_1 \leq \phi \leq \phi_2$, is obtained from this formula by replacing:

- $\frac{\rho_0^3}{3}$ by $\frac{\rho_2^3}{3} - \frac{\rho_1^3}{3} = \frac{\rho_2^2 + \rho_1\rho_2 + \rho_1^2}{3}\Delta\rho$, where $\Delta\rho = \rho_2 - \rho_1$;
- 2π by $\Delta\theta = \theta_2 - \theta_1$;
- $1 - \cos\phi_0$ by $\cos\phi_1 - \cos\phi_2 = \sin\frac{\phi_1 + \phi_2}{2}\frac{\sin\frac{\Delta\phi}{2}}{\frac{\Delta\phi}{2}}\Delta\phi$, where $\Delta\phi = \phi_2 - \phi_1$.

Considering the limits when $\rho_1, \rho_2 \to \rho$, $\theta_1, \theta_2 \to \theta$ and $\phi_1, \phi_2 \to \phi$, we obtain

$$\text{We have } d\mathcal{V} = \rho^2 \sin\phi d\rho d\theta d\phi.$$

Spherical coordinates

Example 1: Use spherical coordinates to compute the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Example 2: Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \; dzdydx$ by changing to spherical coordinates.

Physical applications of integrals

1. Definition of the **density**:
   In two dimensions (For a lamina): $\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$.
   In three dimensions (For a solid): $\rho(x, y, z) = \lim \frac{\Delta m}{\Delta V}$.
2. Formula for the **mass**:
   For a lamina: $m = \iint_{\mathcal{L}} \rho(x, y)\ d\mathcal{A}$.
   For a solid: $m = \iiint_{\mathcal{S}} \rho(x, y, z)\ d\mathcal{V}$.

Physical applications of integrals

1. Formula for the **moment of mass** of a lamina about the two coordinate axes:
   About the $x-$axis: $M_x = \iint_{\mathcal{L}} y\rho(x,y)\ d\mathcal{A}$.
   About the $y-$axis: $M_y = \iint_{\mathcal{L}} x\rho(x,y)\ d\mathcal{A}$.
2. Formula for the **moment of mass** of a solid about the three coordinate planes:
   About the $yz-$plane: $M_{yz} = \iiint_{\mathcal{S}} x\rho(x,y,z)\ d\mathcal{V}$.
   About the $zx-$plane: $M_{zx} = \iiint_{\mathcal{S}} y\rho(x,y,z)\ d\mathcal{V}$.
   About the $xy-$plane: $M_{yz} = \iiint_{\mathcal{S}} z\rho(x,y,z)\ d\mathcal{V}$.
3. Formula for the **center of mass**:
   For a lamina: $\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}$.
   For a solid: $\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{zx}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}$.
   If the density is constant, the center of mass is precisely the **centroid**.

Physical applications of integrals

1. Formula for the **mass moment of inertia** also called the **second moment of mass**:
   For a lamina about the two coordinate axes:

$$I_x = \iint_{\mathcal{L}} y^2 \rho(x,y)\, d\mathcal{A}, \quad I_y = \iint_{\mathcal{L}} x^2 \rho(x,y)\, d\mathcal{A}.$$

   For a solid about the three coordinate axes:

$$I_x = \iiint_{\mathcal{S}} (y^2+z^2)\rho(x,y,z)\, d\mathcal{V}, \quad I_y = \iiint_{\mathcal{S}} (z^2+x^2)\rho(x,y,z)\, d\mathcal{V},$$

$$I_z = \iiint_{\mathcal{S}} (x^2+y^2)\rho(x,y,z)\, d\mathcal{V}.$$

Physical applications of integrals

Example 1: A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

Example 2: Let $\mathcal{S}$ be a solid hemisphere of radius R whose density at any point is proportional to its distance from the center of its base. Find its mass, its center of mass, its moment of inertia about its axis.