Math 203 - Differential and Integral Calculus

Chapter 2: Multiple Integrals

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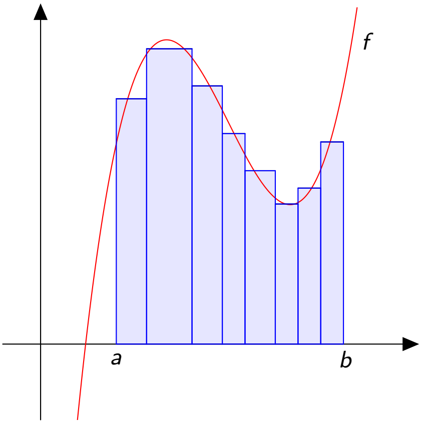
Riemann Sum of a function of one variable

Recall that, for a function $f : [a, b] \to \mathbb{R}$, a partition $\mathcal{P} = \{a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$ of $[a, b]$ and a set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, where $\omega_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, the Riemann sum is defined by

$$\mathcal{R}(f, \mathcal{P}, \Omega) = \sum_{k=1}^{n} f(\omega_k)(x_k - x_{k-1}).$$

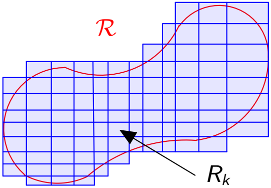
If $f(x) \geq 0$ on $[a, b]$, this is the sum of the areas of the blue rectangles. If f is continuous on $[a, b]$, then the limit



$\lim\limits_{||\mathcal{P}|| \to 0} \mathcal{R}(f, \mathcal{P}, \Omega)$, where $||\mathcal{P}|| = \max(x_k - x_{k-1})$ exists. It is equal by definition to $\displaystyle\int_a^b f(x)dx$, the definite integral of f.

Riemann Sum of a function of two variables

Let $\mathcal{R}$ be a region bounded by a closed curve in $\mathbb{R}^2$. For a function $f : \mathcal{R} \to \mathbb{R}$, consider a partition of $\mathcal{R}$ into small rectangles $\mathcal{P} = \{R_1, R_2, \ldots, R_n\}$ and a set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \subset \mathcal{R}$, where $\omega_k \in R_k$ for $k = 1, \ldots, n$. Let ΔR_k be the area of the rectangles R_k, $k = 1, \ldots, n$, and $||\mathcal{P}|| = \max \Delta R_k$.



The Riemann sum is defined by $\mathcal{R}(f, \mathcal{P}, \Omega) = \sum_{k=1}^{n} f(\omega_k)\Delta R_k$. If $f(x) \geq 0$ on $\mathcal{R}$, this is the sum of the areas of cubes of bases the R_k.

If the limit $\lim_{||\mathcal{P}|| \to 0} \mathcal{R}(f, \mathcal{P}, \Omega)$ exists, we call this limit the double integral of f on $\mathcal{R}$ and denote it $\iint_{\mathcal{R}} f(x, y) d\mathcal{A}$. If f is continuous, then this limit exists.

Properties of the double integral

Let $\mathcal{R}$ be a region bounded by a closed curve in $\mathbb{R}^2$, $\mathcal{P}$ a partitions of $\mathcal{R}$ into rectangles and Ω a set of taken from each rectangle. Notice that for real functions f, f_1, f_2 defined on $\mathcal{R}$ and real number k, we have:

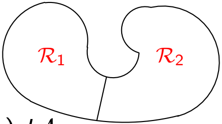
1. $\mathcal{R}(kf, \mathcal{P}, \Omega) = k\mathcal{R}(f, \mathcal{P}, \Omega)$,
2. $\mathcal{R}(f_1 + f_2, \mathcal{P}, \Omega) = \mathcal{R}(f_1, \mathcal{P}, \Omega) + \mathcal{R}(f_2, \mathcal{P}, \Omega)$,
3. If $f \geq 0$, then $\mathcal{R}(f, \mathcal{P}, \Omega) \geq 0$.

We deduce the following

1. $\displaystyle\iint_{\mathcal{R}} kf(x,y)d\mathcal{A} = k\iint_{\mathcal{R}} f(x,y)d\mathcal{A}$,
2. $\displaystyle\iint_{\mathcal{R}} (f_1(x,y) + f_2(x,y))d\mathcal{A} = \iint_{\mathcal{R}} f_1(x,y)d\mathcal{A} + \iint_{\mathcal{R}} f_2(x,y)d\mathcal{A}$,
3. If $f \geq 0$, then $\displaystyle\iint_{\mathcal{R}} f(x,y)d\mathcal{A} \geq 0$.

Properties of the double integral

4. If a curve divides a region $\mathcal{R}$ into two subregions, $\mathcal{R}_1, \mathcal{R}_2$, then



$$\iint_{\mathcal{R}} f(x,y)d\mathcal{A} = \iint_{\mathcal{R}_1} f(x,y)d\mathcal{A} + \iint_{\mathcal{R}_2} f(x,y)d\mathcal{A}.$$

Iterated double integral:

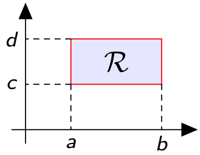
Usually, if we apply a simple integral twice, we shorten the notation by omitting the brakets.

We write $\int_a^b \int_c^d f(x,y)dxdy$ instead of $\int_a^b \left(\int_c^d f(x,y)dx \right) dy.$

This is called an iterated double integral. To compute a double integral, most of the times we use iterated double integrals.

Double integral over a rectangular region

Theorem (Fubini)



If $\mathcal{R} = [a, b] \times [c, d]$ *is a rectangular region and* $f : \mathcal{R} \to \mathbb{R}$ *is continuous, then*

$$\iint_{\mathcal{R}} f(x, y) d\mathcal{A} = \int_c^d \int_a^b f(x, y) dxdy = \int_a^b \int_c^d f(x, y) dydx.$$

Example 1: Compute $\iint_{[0,1]\times[0,2]} \left(x^2 + xy - y^3\right) d\mathcal{A}$.

Solution: We have

$$\iint_{[0,1]\times[0,2]} \left(x^2 + xy - y^3\right) d\mathcal{A} = \int_0^2 \int_0^1 \left(x^2 + xy - y^3\right) dxdy$$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2} - xy^3\right]_0^1 dy = \int_0^2 \left(\frac{1}{3} + \frac{y}{2} - y^3\right) dy = \left[\frac{y}{3} + \frac{y^2}{4} - \frac{y^4}{4}\right]_0^2 = -\frac{7}{3}.$$

Double integral over a rectangular region

Another method: We have

$$\iint_{[0,1]\times[0,2]} \left(x^2+xy-y^3\right) d\mathcal{A} = \int_0^1\int_0^2 \left(x^2+xy-y^3\right) dydx$$

$$= \int_0^1 \left[x^2y+\frac{xy^2}{2}-\frac{y^4}{4}\right]_0^2 dx = \int_0^1 \left(2x^2+2x-4\right)dy = \left[\frac{2x^3}{3}+x^2-4x\right]_0^1 = -\frac{7}{3}.$$

Example 2: Compute $\displaystyle\iint_{[0,1]\times[-1,1]} \frac{1}{y^3}e^{\frac{x}{y}}\, d\mathcal{A}$.

Solution: We have

$$\iint_{[0,1]\times[-1,1]} \frac{1}{y^3}e^{\frac{x}{y}}\, d\mathcal{A} = \int_{-1}^1\int_0^1 \frac{1}{y^3}e^{\frac{x}{y}}\, dxdy = \int_{-1}^1 \left[\frac{1}{y^2}e^{\frac{x}{y}}\right]_0^1 dy$$

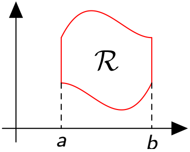
$$= \int_{-1}^1 \left(\frac{1}{y^2}e^{\frac{1}{y}}-\frac{1}{y^2}\right) dy = \left[-e^{\frac{1}{y}}+\frac{1}{y}\right]_{-1}^1 = 2\sinh 1+2.$$

Algebraically, we don't know how to compute this double integral when starting first by integrating with respect to y.

Double integral over a non rectangular region

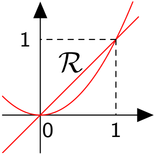
Theorem

Let $f : \mathcal{R} \to \mathbb{R}$ *be a continuous function defined on the region* $\mathcal{R} = \{(x, y) | a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$. *We have* $\displaystyle\iint_{\mathcal{R}} f(x, y) d\mathcal{A} = \int_a^b \int_{\phi(x)}^{\phi_2(x)} f(x, y) dy dx.$



Example 1: Compute $\displaystyle\iint_{\mathcal{R}} x^2 y \; d\mathcal{A}$, where $\mathcal{R}$ is the region bounded by the curves $y = x$ and $y = x^2$.

Solution: The intersection points of the two curves are the solutions of the system $\begin{cases} y &= x \\ y &= x^2 \end{cases}$. These are the points $(0, 0)$ and $(1, 1)$.

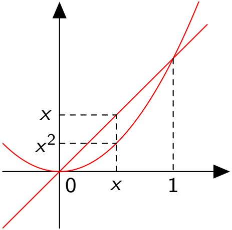


Double integral over a non rectangular region

First method: We have

$$\iint_{\mathcal{R}} x^2y\ d\mathcal{A} = \int_0^1 \int_{x^2}^{x} x^2y\ dydx = \int_0^1 \left[\frac{x^2y^2}{2}\right]_{x^2}^{x}\ dx$$

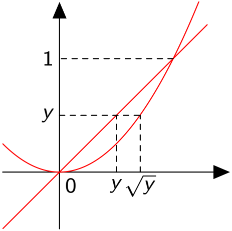
$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2}\right) dx = \left[\frac{x^5}{10} - \frac{x^7}{14}\right]_0^1 = \frac{1}{35}.$$



Second method: We have

$$\iint_{\mathcal{R}} x^2y\ d\mathcal{A} = \int_0^1 \int_{y}^{\sqrt{y}} x^2y\ dxdy = \int_0^1 \left[\frac{x^3y}{3}\right]_{y}^{\sqrt{y}}\ dy$$

$$= \int_0^1 \left(\frac{y^{\frac{5}{2}}}{3} - \frac{y^4}{3}\right) dx = \left[\frac{2y^{\frac{7}{2}}}{21} - \frac{y^5}{15}\right]_0^1 = \frac{1}{35}.$$



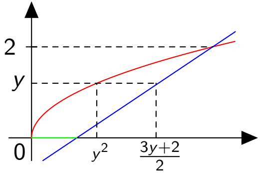
Areas and volumes

Theorem

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a bounded region and $f : \mathcal{R} \to \mathbb{R}$ a nonnegative integrable function.

1. *The area of the region $\mathcal{R}$ is given by $Area(\mathcal{R}) = \iint_{\mathcal{R}} 1\ d\mathcal{A}$.*
2. *The volume of the solid over $\mathcal{R}$ bounded above by the surface given by f is given by the formula $Volume = \iint_{\mathcal{R}} f(x, y)\ d\mathcal{A}$.*

Example: Find the area of the region $\mathcal{R}$ bounded by the curves of equations $y = \sqrt{x}$, $y = 0$ and $2x - 3y = 2$.
Solution: We compute $\iint_{\mathcal{R}} 1\ d\mathcal{A}$ in two ways:



1. **First way:** $\displaystyle\iint_{\mathcal{R}} 1\ d\mathcal{A} = \int_0^2 \int_{y^2}^{\frac{3y+2}{2}} 1 dxdy =$

$$\int_0^2 \left(\frac{3y+2}{2} - y^2\right) dy = \left[\frac{3y^2}{4} + y - \frac{y^3}{3}\right]_0^2 = \frac{7}{3}.$$

Areas and volumes

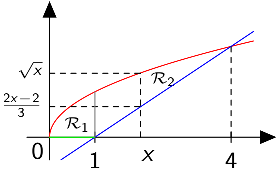
2. **Second way:**

$$\iint_{\mathcal{R}} 1 \; d\mathcal{A} = \iint_{\mathcal{R}_1} 1 \; d\mathcal{A} + \iint_{\mathcal{R}_2} 1 \; d\mathcal{A}$$

$$= \int_0^1 \int_0^{\sqrt{x}} 1 dy dx + \int_1^4 \int_{\frac{2x-2}{3}}^{\sqrt{x}} 1 dy dx$$

$$= \int_0^1 \sqrt{x} dx + \int_1^4 \left(\sqrt{x} - \frac{2x-2}{3} \right) dx = \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^1 + \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2 - 2x}{3} \right]_1^4$$

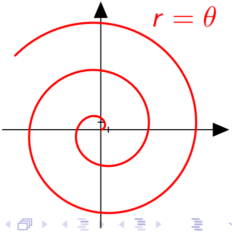
$$= \frac{2}{3} + \left(\frac{16}{3} - \frac{8}{3} \right) - \left(\frac{2}{3} + \frac{1}{3} \right) = \frac{7}{3}.$$



Double integrals in polar coordinates

Recall that the relation between polar coordinates (r, θ) and Cartesian coordinates is given by $x = r\cos\theta, y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$. Here are examples of equations of some curves in polar coordinates:

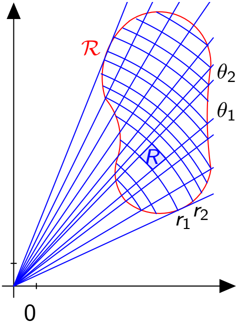
1. The parabola $y = x^2$ in polar coordinates: $r\sin\theta = r^2\cos^2\theta$ which becomes $r = \sec\theta\tan\theta$.
2. The line $x = 1$ in polar coordinates: $r\cos\theta = 1 \Leftrightarrow r = \sec\theta$.
3. The circle $x^2 + y^2 = 1$ in polar coordinates becomes $r = 1$.
4. The curve $r = \sin\theta$ becomes in Cartesian coordinates $x^2 + y^2 = y$, which is equivalent to $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$, which is a circle.
5. The equation of the spiral $r = \theta$ is written in Cartesian coordinates
$x = \sqrt{x^2 + y^2}\cos\sqrt{x^2 + y^2}$,
$y = \sqrt{x^2 + y^2}\sin\sqrt{x^2 + y^2}$.



Some curves have equation in polar coordinates much simpler than in Cartesian coordinates.

Double integrals in polar coordinates

In Cartesian coordinates, the region $\mathcal{R}$ is decomposed into small rectangles whose area is $\Delta x \Delta y$. This is why we have $d\mathcal{A} = dxdy$. However, in polar coordinates, $\mathcal{R}$ is decomposed into small polar rectangles $R = \{(r, \theta) | r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$. To compute the area of the polar triangle, recall that the area of a disc or radius r is πr^2, a half disc $\frac{\pi}{2} r^2$ and a sector of radius r and angle $\Delta\theta = \theta_2 - \theta_1$ is $\frac{\Delta\theta \cdot r^2}{2}$. Therefore, the area of the polar rectangle R is



$\Delta R = \frac{\Delta\theta}{2}\left(r_2^2 - r_1^2\right) = \left(r_1 + \frac{\Delta r}{2}\right)\Delta r \Delta\theta \approx r_1 \Delta r \Delta\theta$, where $\Delta r = r_2 - r_1$.

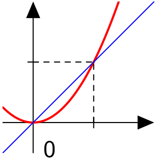
We deduce that in polar coordinates $d\mathcal{A} = r dr d\theta$.

Example 1: Compute the double integral of $f : \mathcal{R} \to \mathbb{R}, (r, \theta) \mapsto r^2 \cos\theta$, where $\mathcal{R}$ is the region bounded by $r = \sin\theta$. **Solution:** We have

$$\iint_{\mathcal{R}} r^2 \cos\theta d\mathcal{A} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sin\theta} r^3 \cos\theta dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \frac{\sin^4\theta}{4} d\theta = \left[\frac{\sin^5\theta}{20}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{10}.$$

Double integrals in polar coordinates

Example 2: Compute in Cartesian coordinates and in polar coordinates the area of the region $\mathcal{R}$ bouded by the parabola $y = x^2$ and the line $y = x$.



Solution: We have

1. In Cartesian coordinates:
$$\iint_{\mathcal{R}} 1 d\mathcal{A} = \int_0^1 \int_{x^2}^{x} 1 dydx = \int_0^1 (x - x^2) dydx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$
2. In polar coordinates: $\iint_{\mathcal{R}} 1 d\mathcal{A} = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta\tan\theta} r dr d\theta =$
$$\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sec\theta\tan\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2\theta\tan^2\theta d\theta = \frac{1}{6}\left[\tan^3\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{6}.$$