Math 203 - Differential and Integral Calculus

Chapter 1: Sequences and Series

Dr. Malik Talbi
King Saud University, Mathematic's Department

January 30, 2025

Definition of sequences

A numerical sequence is a list of numbers:

- Finite sequence: $2; \frac{1}{2}; 3; -5; \sqrt{7}; e$.
- Infinite sequence: $1; -5; \pi; 19; \frac{7}{3}; \dots$

Definition

An **infinite sequence** or simply a **sequence** is a function defined on natural numbers. We denote it $\{a_n\}$.

The **terms** of the sequence are: $a_1; a_2; a_3; \dots$.

In the example above:

$$a_1 = 1;\ a_2 = -5;\ a_3 = \pi;\ a_4 = 19;\ a_5 = \frac{7}{3}; \dots.$$

a_n is called the **general term** of the sequence.

Examples of sequences

Examples: $\left\{\dfrac{n(n+1)}{2}\right\}$; $\{\cos(n\pi)\}$; $\left\{\cos\left(\dfrac{n\pi}{2}\right)\right\}$; $\left\{\dfrac{n+1}{n}\right\}$.

1. For the sequence of general term $a_n = \frac{n(n+1)}{2}$, we have
$$a_1 = 1; a_2 = 3; a_3 = 6; a_4 = 10; a_5 = 15; \ldots$$
2. For the sequence of general term $a_n = \cos(n\pi)$, we have
$$a_1 = -1; a_2 = 1; a_3 = -1; a_4 = 1; a_5 = -1; \ldots$$
Notice that here the general term can be simply written $a_n = (-1)^n$.
3. For the sequence of general term $a_n = \cos\left(\frac{n\pi}{2}\right)$, we have
$$a_1 = 0; a_2 = -1; a_3 = 0; a_4 = 1; a_5 = 0; \ldots$$
4. For the sequence of general term $a_n = \frac{n+1}{n}$, we have
$$a_1 = 2; a_2 = \frac{3}{2}; a_3 = \frac{4}{3}; a_4 = \frac{5}{4}; a_5 = \frac{6}{5}; \ldots$$

Sequences defined by their first terms

Remark: Sometimes, sequences are given by their first terms following an implicite rule. Sometimes, a closed form of the general term can be deduced and others no.

Examples:

- A closed form of the general term of the sequence $1; 1; 2; 2; 3; 3; 4; 4; \ldots$ can be given by

$$a_n = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ \frac{n}{2} & \text{if } n \text{ is even,} \end{cases} \quad \text{or equivalently } a_n = \left\lceil \frac{n}{2} \right\rceil.$$

  The ceiling of x is defined by $\lceil x \rceil = m$, where m is an integer satisfying $m - 1 < x \leq m$.

- For the sequence $1; 4; 9; 16; 25; 36; \ldots$, $a_n = n^2$.
- For the sequence of prime numbers $2; 3; 5; 7; 11; 13; 17; 19; \ldots$, we don't know a closed formula for the general term a_n.

Limits of sequences

Remark: To move from an integer to another integer, we need to jump. We cannot approach integers continuously from other integers. That is why, for sequences, when we talk about limits we mean $n \to \infty$ and write simply $\lim a_n$.

Definition

- Let $L \in \mathbb{R}$. We say that $\lim a_n = L$, if for any $\epsilon > 0$, there exists an natural number N such that for all $n \geq N$, we have $|a_n - L| < \epsilon$.
- We say that $\lim a_n = \infty$, if for any $A > 0$, there exists an natural number N such that for all $n \geq N$, we have $a_n \geq A$.
- We say that $\lim a_n = -\infty$, if for any $B < 0$, there exists an natural number N such that for all $n \geq N$, we have $a_n \leq B$.
- We say that $\lim a_n$ does not exist if none of the properties above is satisfied.

Examples of limits of sequences

Examples:

- $\lim \frac{n+1}{n} = \lim \left(1 + \frac{1}{n}\right) = 1.$
- $\lim \frac{3n^3+n^2-n+1}{2n^3-n+2} = \lim \frac{3n^3}{2n^3} = \lim \frac{3}{2} = \frac{3}{2}.$
- $\lim \frac{3n^3+n^2-n+1}{n^2-n+2} = \lim \frac{3n^3}{n^2} = \lim 3n = \infty.$
- The limits of the sequences $(-1)^n$, $\cos\left(\frac{n\pi}{2}\right)$, $\sin\left(\frac{n\pi}{30}\right)$ do not exist. In general, any periodic nonconstant sequence has no limit.

To use the l'Hospital's Rule, we need the following theorem:

Theorem

If $a_n = f(n)$, for some function f defined on $[1, \infty)$, and $\lim_{x\to\infty} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim a_n = L$.

Using l'Hospital's rule

Examples:

1. For $a_n = \frac{\ln n}{n}$, consider the function defined on $[1, \infty)$ by $f(x) = \frac{\ln x}{x}$. We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$. Therefore, $\lim a_n = 0$.
2. For $a_n = \frac{e^n}{n}$, consider the function defined on $[1, \infty)$ by $f(x) = \frac{e^x}{x}$. We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$. Therefore, $\lim a_n = \infty$.

What happens if $\lim_{x\to\infty} f(x)$ does not exist:

> *Consider* $f(x) = \sin(\pi x)$. *Because* $f(x)$ *is a periodic function,* $\lim_{x\to\infty} f(x)$ *does not exist. However, for the sequence defined by* $a_n = f(n)$, *we have* $a_n = 0$ *for all* n *and* $\lim a_n = 0$. *We deduce the following:*

If $\lim f(x)$ does not exist, we cannot deduce nothing about $\lim a_n$.

Convergent and divergent sequences

Definition

We say that a sequence is **convergent** if it has a finite limit. Otherwise, it is **divergent**.

Theorem

Let $\{a_n\}; \{b_n\}$ *be two sequences and* f *a real function such that* $a_n = f(b_n)$. *If* $\lim b_n = \ell \in \mathbb{R} \cup \{\pm\infty\}$ *and* $\lim_{x \to \ell} f(x) = L$, *then* $\lim a_n = L$.
In particular, if f *is continuous at* ℓ, *then* $\lim a_n = f(\ell)$.

Example: Consider the sequence $\left\{n^{\frac{1}{n}}\right\}$. We have $n^{\frac{1}{n}} = e^{\frac{\ln n}{n}}$. But we have already seen that $\lim \dfrac{\ln n}{n} = 0$. Because $x \mapsto e^x$ is continuous at 0, we deduce that $\lim n^{\frac{1}{n}} = \lim e^{\frac{\ln n}{n}} = e^0 = 1$.

Convergent and divergent sequences

Other examples:

- Consider the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$.

  $$\text{We have } \left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} = e^{\frac{\ln(1+1/n)}{1/n}}.$$

  Consider $f(x) = \frac{\ln(1+x)}{x}$. Because $\lim \frac{1}{n} = 0$, we compute

  $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1, \text{ by l'Hôpital's rule.}$$

  Because $x \mapsto e^x$ is continuous at 1, we deduce that $\lim\left(1+\frac{1}{n}\right)^n = e^1 = e$.

- In the same way, one can prove that $\lim\left(1+\frac{x}{n}\right)^n = e^x$.

Comparison between sequences

Theorem

Let $\{a_n\}, \{b_n\}$ be two sequences for which there exists N such that for all $n \geq N$, $a_n = b_n$. We have $\lim a_n = \lim b_n$.

This theorem means that to compute a limit of a sequence, it doesn't matter what happens for the first terms. We have to focus only on last terms.

Example: Consider the sequence of general term

$$a_n = \frac{n^2+n+1}{2n^2-n+2} + \left\lfloor \frac{1000}{n} \right\rfloor \left(\frac{n! \cos(n^n) + n^n}{1 + n \ln n} \right).$$

We want to comput $\lim a_n$. We notice that for all $n > 1000$, we have $0 \leq \frac{1000}{n} < 1$ and therefore, $\left\lfloor \frac{1000}{n} \right\rfloor = 0$. Hence, for all $n > 1000$, $a_n = \dfrac{n^2+n+1}{2n^2-n+2}$ and $\lim a_n = \frac{1}{2}$.

Comparison between sequences

Theorem

Let $\{a_n\}, \{b_n\}$ be two sequences such that $a_n \leq b_n$ for all n.

1. *If $\lim a_n = \infty$, then $\lim b_n = \infty$.*
2. *If $\lim b_n = -\infty$, then $\lim a_n = -\infty$.*
3. *If $\{c_n\}$ is a third sequence such that $a_n \leq b_n \leq c_n$ for all n and $\lim a_n = \lim c_n = l$, then $\lim b_n = l$ (Sandwich theorem).*

Examples:

1. Consider the sequence of general term $a_n = n^3 + (-1)^n n^2$. We have $a_n \geq n^3 - n^2$. But $\lim \left(n^3 - n^2\right) = \infty$. We deduce that $\lim a_n = \infty$.
2. Consider the sequence of general term $a_n = \dfrac{\cos n}{n}$. Because $-1 \leq \cos n \leq 1$ for all n, we have $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$. But $\lim -\frac{1}{n} = \lim \frac{1}{n} = 0$. We deduce by Sandwich theorem that

Monotonic sequences

Definition

1. A sequence $\{a_n\}$ is said to be **increasing** if for any $n \geq 1$, we have $a_{n+1} > a_n$.
2. A sequence $\{a_n\}$ is said to be **non-decreasing** if for any $n \geq 1$, we have $a_{n+1} \geq a_n$.
3. A sequence $\{a_n\}$ is said to be **decreasing** if for any $n \geq 1$, we have $a_{n+1} < a_n$.
4. A sequence $\{a_n\}$ is said to be **non-increasing** if for any $n \geq 1$, we have $a_{n+1} \leq a_n$.
5. A sequence is said to be **monotonic** if it satisfies one of the four conditions above.

Remark: Any increasing sequence is non-decreasing. Any decreasing sequence is non-increasing. But the converse is not true.

Monotonic sequences

Examples:

- The sequence $\{e^n\}$ is increasing.
- The sequence $1; 1; 2; 2; 3; 3; 4; 4; 5; 5; \ldots$ is non-decreasing. However it is not increasing.
- The sequence $\{\frac{1}{n}\}$ is decreasing.
- The sequence $\left\{\left\lfloor \frac{1000}{n} \right\rfloor\right\}$ is non-increasing.

Theorem

1. *Any monotonic sequence has a limit in* $\mathbb{R} \cup \{\pm\infty\}$.
2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences satisfying* $a_n \leq b_n$ *for all* $n \geq 1$.
    1. *If* $\{a_n\}$ *is non-decreasing and* $\{b_n\}$ *is convergent, then* $\{a_n\}$ *is convergent.*
    2. *If* $\{b_n\}$ *is non-increasing and* $\{a_n\}$ *is convergent, then* $\{b_n\}$ *is convergent.*

Bouned monotonic sequences

Example: Consider the two sequences defined by

$$a_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots ;$$

$$b_n = 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1)n} + \cdots .$$

The sequence $\{a_n\}$ is increasing since $a_{n+1} = a_n + \frac{1}{(n+1)!} > a_n$ for all $n \geq 1$. Moreover, for all $n \geq 1$, we have $a_n \leq b_n$. On the other hand, because $\frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}$, we have

$$b_n = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n}.$$

This implies that $\{b_n\}$ is convergent with $\lim b_n = 2$.
We deduce that $\{a_n\}$ is convergent.

Bouned monotonic sequences

Definition

A sequence $\{a_n\}$ is said to be **bounded**, if there exist two real numbers A and B such that $A \leq a_n \leq B$ for all $n \geq 1$.

As a consequence of the previous theorem:

Theorem

Any bounded monotonic sequence is convergent.

For the sequence defined by $a_1 = 1$ and $\forall n \geq 1$, $a_{n+1} = 2a_n - \dfrac{a_n^2}{2}$.
We have $0 \leq a_1 \leq 2$. Assume $0 \leq a_n \leq 2$ for $n \geq 1$. We have
$0 \leq \dfrac{a_n(4 - a_n)}{2} = a_{n+1} = 2 - \dfrac{(a_n - 2)^2}{2} \leq 2$.
Moreover, $a_{n+1} - a_n = \dfrac{a_n(2 - a_n)}{2} \geq 0$ for all $n \geq 1$, ie $a_{n+1} \geq a_n$.
Therefore, $\{a_n\}$ is bounded and monotonic. It is convergent.

Geometric sequences

Definition

A **geometric sequence** $\{a_n\}$ is a sequence of non-zero terms with constant ratio $r = \frac{a_{n+1}}{a_n}$. The constant r is called the **ratio** of the sequence.

More precisely, the sequence has the form:

$$a_1; a_1 r; a_1 r^2; a_1 r^3; \ldots; a_1 r^{n-1}; \ldots.$$

Examples:

- The sequence $3; 6; 12; 24; 48; 96; 192; \ldots$ where $a_n = 3 \times 2^{n-1}$ is divergent with $\lim a_n = \infty$.
- The sequence $16; 8; 4; 2; 1; \frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$ where $a_n = 16 \times \left(\frac{1}{2}\right)^{n-1} = \frac{8}{2^n}$ is convergent with $\lim a_n = 0$.

Geometric sequences

Theorem

If $\lim |a_n| = 0$, *then* $\lim a_n = 0$.

- The sequence $16; -8; 4; -2; 1; -\frac{1}{2}; \frac{1}{4}; -\frac{1}{8}; \ldots$ where $a_n = 16 \times \left(-\frac{1}{2}\right)^{n-1}$ is convergent with $\lim a_n = 0$.

In general, we have

Theorem

Let $\{a_n\}$ *be a geometric sequence of ratio* r.

1. *If* $|r| > 1$, *then* $\{a_n\}$ *is divergent.*
2. *If* $|r| ><$, *then* $\{a_n\}$ *is convergent.*
3. *If* $r = 1$, *then* $\{a_n\}$ *is constant and therefore convergent.*
4. *If* $r = -1$, *then* $\{a_n = (-1)^{n-1} a_1\}$ *is divergent.*

Series

Definition

Let $\{a_n\}$ be a sequence. The **series** of terms a_n is the "formal" sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

It is denoted $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.

Examples:

- $0.333333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots + \frac{3}{10^n} + \cdots .$
- $1 + 2 + 3 + 4 + \cdots + n + \cdots .$
- $1 + (-1) + 1 + (-1) + \cdots + (-1)^{n+1} + \cdots .$

Partial sums

Given a sequence $\{a_n\}$, we consider the sequence of **partial sums** defined by:

$$\begin{aligned} s_1 &= a_1; \\ s_2 &= a_1 + a_2; \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n; \\ &\vdots \end{aligned}$$

The general term of some partial sums:

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$;
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$;
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$;
4. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)\times n} = 1 - \frac{1}{n}$.

Partial sums

For the sum $s_n = 1 + 2 + 3 + \cdots + n$, we have

$$\begin{array}{rccccccccc} s_n = & 1 & + & 2 & + & 3 & + \cdots + & n \\ s_n = & n & + & (n-1) & + & (n-2) & + \cdots + & 1 \\ \hline 2s_n = & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) \end{array}$$

This gives, $s_n = \frac{n(n+1)}{2}$.
A second method: we have

$$\begin{aligned} (n+1)^2 &= (1^2 - 0^2) + (2^2 - 1^2) + \cdots + ((n+1)^2 - n^2) \\ &= \sum_{k=0}^{n} ((k+1)^2 - k^2) = \sum_{k=0}^{n} (2k+1) \\ &= 2\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 = 2s_n + n + 1. \end{aligned}$$

Therefore, $2s_n = (n+1)^2 - n - 1 = n^2 + n$ and $s_n = \frac{n(n+1)}{2}$.

Partial sums

- For the sum $s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$, we have

$$\begin{aligned}(n+1)^3 &= (1^3 - 0^3) + (2^3 - 1^3) + \cdots + \left((n+1)^3 - n^3\right) \\ &= \sum_{k=0}^{n} \left((k+1)^3 - k^3\right) = \sum_{k=0}^{n}(3k^2 + 3k + 1) \\ &= 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 = 3s_n + \frac{3n(n+1)}{2} + (n+1).\end{aligned}$$

  Therefore, $s_n = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} = \frac{(n+1)\left(2(n+1)^2 - 3n - 2\right)}{6} = \frac{n(n+1)(2n+1)}{6}$.

- Do the same for the sum $s_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$ to obtain $s_n = \frac{n^2(n+1)^2}{4}$.

Partial sums

Conversely, given the sequence $\{s_n\}$ of partial sums, we can find back the sequence $\{a_n\}$ by

$$a_1 = s_1,\ a_2 = s_2 - s_1, \ldots, a_n = s_n - s_{n-1}, \ldots.$$

Examples:

- For $s_n = n^2$, we have $a_1 = 1$ and $a_n = n^2 - (n-1)^2 = 2n - 1$ for $n \geq 2$.
- For $s_n = \frac{1}{n}$, we have $a_1 = 1$ and $a_n = \frac{1}{n} - \frac{1}{n-1} = -\frac{1}{n(n-1)}$ for $n \geq 2$.
- For $s_n = (-1)^n$, we have $a_1 = -1$ and $a_n = (-1)^n - (-1)^{n-1} = 2(-1)^n$ for $n \geq 2$.
- For $s_n = \ln n$, we have $a_1 = 0$ and $a_n = \ln n - \ln(n-1) = \ln \frac{n}{n-1}$ for $n \geq 2$.

Convergent series

Definition

1. If the sequence of partial sums $\{s_n\}$ is convergent and $s = \lim s_n$, we say that the series $\sum a_n$ is convergent, s is the sum of the series and write $\displaystyle\sum_{n=1}^{\infty} a_n = s$.
2. If the sequence of partial sums $\{s_n\}$ is divergent, we say that the series $\sum a_n$ is divergent.

Examples:

- $0.333333\cdots = \sum \frac{3}{10^n} = \frac{1}{3}$ is convergent.
- $\sum n = \infty$ is divergent.
- $\sum (-1)^n$ is divergent.

Geometric series

For the sum $s_n = 1 + r + r^2 + \cdots + r^{n-1}$, we have

$$rs_n = r + r^2 + r^3 + \cdots + r^n.$$

Therefore $(1-r)s_n = s_n - rs_n = 1 - r^n$. Hence

$$1 + r + r^2 + \cdots + r^{n-1} = \begin{cases} \frac{1-r^n}{1-r}, & \text{if } r \neq 1; \\ n, & \text{if } r = 1. \end{cases}$$

Theorem

Let $\{a_n\}$ be a geometric sequence of ratio r. We have

1. *If $|r| < 1$, then the series $\sum a_n$ is convergent with sum $\dfrac{a_1}{1-r}$.*
2. *If $|r| \geq 1$, then the series $\sum a_n$ is divergent.*

The $n^{th}-$term test

Theorem

If the series $\sum a_n$ *is convergent, then* $\lim a_n = 0$.

Indeed, if $s = \lim s_n$, we have

$$\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = s - s = 0.$$

This means that if $\lim a_n \neq 0$, then the series $\sum a_n$ is divergent. This is called the $n^{th}-$**term test** for a divergence of a series.
Examples: The series $\sum \ln n$, $\sum (-1)^n$, $\sum \frac{n-1}{n+1}$ are all divergent, directly from the $n^{th}-$term test.

Remark

If $\lim a_n = 0$, *this is* **not enough to deduce** *that the series* $\sum a_n$ *converges.*

Example: $\lim \frac{1}{n} = 0$, but $\sum \frac{1}{n}$ is divergent as we will see.

Comparision of series

Theorem

1. *If we delete the first $N-1$ terms of a series $\sum a_n$ then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=N}^{\infty} a_n$ are of the same nature.*
2. *If the series $\sum a_n$ and $\sum b_n$ satisfy $a_n = b_n$ for all $n \geq N$, for some natural number N, then they are of the same nature.*

Examples:

1. The series $\sum a_n$, where $a_n = n$, for $n \leq 10^6$, and $a_n = \frac{1}{n(n-1)}$, for $n > 10^6$, is convergent since $\sum_{n=10^6+1}^{\infty} = \frac{1}{10^6}$ is convergent.
2. The series $\sum a_n$ and $\sum b_n$, where $b_n = a_n + \left\lceil \frac{1000}{n} \right\rceil$ are of the same nature since $a_n = b_n$ for all $n > 1000$.

Series of non-negative terms

Recall that for a series $\sum a_n$, the sequence of partial sums $\{s_n\}$ satisfies $a_n = s_n - s_{n-1}$ for all $n \geq 2$. Therefore, the general term a_n of the series is non-negative means that the sequence $\{s_n\}$ is non-decreasing. Hence, all theorems for non-decreasing sequences can be applied to series of non-negative terms.

Theorem

Let $\sum a_n$ be a series of non-negative terms. If there exists M such that $\sum_{k=1}^{n} a_k \leq M$ for all $n \geq 1$, then the series $\sum a_n$ is convergent.

Example: Consider the series $\sum \frac{1}{n!}$. For all $n \geq 2$, we have

$$\sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{n} < 2.$$

We deduce that the series $\sum 1/n!$ of positive terms is convergent

Comparision of series of non-negative terms

Theorem

Let $\sum a_n$ and $\sum b_n$ be two series of non-negative terms such that $a_n \leq b_n$ for all $n \geq 1$.

1. *If the series $\sum b_n$ is convergent, then the series $\sum a_n$ is convergent.*
2. *If the series $\sum a_n$ is divergent, then the series $\sum b_n$ is divergent.*

1. Let $p \geq 2$. We have $0 < \frac{1}{n^p} \leq \frac{1}{n^2} < \frac{1}{n(n-1)}$ for all $n \geq 2$. But the series of positive terms $\sum \frac{1}{n(n-1)} = 1$ is convergent. We deduce that the series of positive terms $\sum \frac{1}{n^p}$ is convergent.
2. Let $1 \geq p > 0$. We have $0 < \frac{1}{n} \leq \frac{1}{n^p}$ for all $n \geq 1$. But the series of positive terms $\sum \frac{1}{n}$ is divergent. We deduce that the series of positive terms $\sum \frac{1}{n^p}$ is divergent.

The harmonic series

Definition

We call the series $\sum \frac{1}{n}$, the **harmonic** series.

To prove that the harmonic series is divergent, consider the series of general term $a_n = \frac{1}{2^k}$, for $2^{k-1} < n \leq 2^k$. That is $a_1 = 1$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{4}$; $a_4 = \frac{1}{4}$; $a_5 = \frac{1}{8}$; $a_6 = \frac{1}{8}; \ldots$. We have $\frac{1}{n} \geq a_n$ for all $n \geq 1$ and then $\displaystyle\sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} a_k$. On the other hand

$$\sum_{n=1}^{2^k} a_n = 1 + \tfrac{1}{2} + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2\times} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4\times} + \cdots + \underbrace{\frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k}}_{2^{k-1}\times}$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k\times} = 1 + \tfrac{k}{2}.$$ Therefore $\sum a_n$ is divergent.

The integral test

Theorem

Let $\sum a_n$ be a series of non-negative terms for which there exists a function f defined on $[1, \infty)$ such that $a_n = f(n)$ for all $n \geq 1$. If the function f is non-increasing and continuous, then the series $\sum a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ are of the same nature (Either they are both convergent, or both divergent).

Because f is continuous and non-increasing, it is integrable on any closed interval and we have $\int_{n-1}^{n} f(x)dx \leq a_n \leq \int_{n}^{n+1} f(x)dx$ for all $n \geq 2$. We deduce that $\int_1^{\infty} f(x)dx \leq \sum_{n=2}^{\infty} a_n \leq \int_2^{\infty} f(x)dx$.

Hence, the series $\sum a_n$ of non-negative terms and the improper integral of the non-negative function f are of the same nature.

Application of the integral test

Theorem

For $\alpha \geq 0$, consider the series $\sum \frac{1}{n^\alpha}$ of positive terms.

1. *If $\alpha > 1$, the series is convergent.*
2. *If $1 \geq \alpha \geq 0$, the series is divergent.*

Examples:

1. The series $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, $\sum \frac{1}{n\sqrt{n}}$, $\sum \frac{1}{n^{1.1}}$ are convergent.
2. The series $\sum \frac{1}{n}$, $\sum \frac{1}{\sqrt{n}}$, $\sum \frac{1}{\sqrt[3]{n}}$, $\sum \frac{1}{n^{0.9}}$ are divergent.

Application of the integral test to the series $\sum \frac{1}{n^\alpha}$

Proof of the theorem: Consider the function defined on $[1, \infty)$ by $f(x) = \frac{1}{x^\alpha}$. The function f is positive and continuous. Moreover, because $f'(x) = -\frac{\alpha}{x^{\alpha+1}} < 0$ for all $x \geq 1$, the function f is decreasing. We can apply the integral test.

If $\alpha \neq 1$, we have $\int_1^\infty \frac{1}{x^\alpha} dx = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^\infty$.

1. If $\alpha > 1$, the improper integral $\int_1^\infty \frac{1}{x^\alpha} dx = \frac{1}{1-\alpha}$ is convergent, and therefore, the series $\sum \frac{1}{n^\alpha}$ is convergent.
2. If $0 \leq \alpha < 1$, the improper integral $\int_1^\infty \frac{1}{x^\alpha} dx = \infty$ is divergent, and therefore, the series $\sum \frac{1}{n^\alpha}$ is divergent.
3. If $\alpha = 1$, the improper integral $\int_1^\infty \frac{1}{x} dx = [\ln x]_1^\infty = \infty$ is divergent, and therefore, the harmonic series $\sum \frac{1}{n}$ is divergent.

Exercise: Prove that the series $\sum \frac{1}{n \ln^\alpha n}$ is convergent for $\alpha > 1$ and divergent for $0 \leq \alpha \leq 1$.

Another example for the integral test

Exercise: Prove that the series $\sum \frac{n^2}{e^{n^3}}$ is convergent.

Solution: Consider the function f defined on $[1, \infty)$ by $f(x) = \frac{x^2}{e^{x^3}}$.
The function f is positive and continuous.
Moreover, we have $f'(x) = \frac{2xe^{x^3} - 3x^4 e^{x^3}}{\left(e^{x^3}\right)} = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for all $x \geq 1$. Therefore, the function f is decreasing.
We can apply the integral test. We have

$$\int_1^{\infty} \frac{x^2}{e^{x^3}} dx = \left[-\frac{1}{3} e^{-x^3}\right]_1^{\infty} = \frac{1}{3e}$$

is convergent.
We deduce that the series $\sum \frac{n^2}{e^{n^3}}$ is convergent.

Comparision of series of positive terms

Theorem

Let $\sum a_n$ and $\sum b_n$ be two series of positive terms.

1. *If there exist m and M and N such that for all $n \geq N$, $0 < m \leq \frac{a_n}{b_n} \leq M$, then the series $\sum a_n$ and $\sum b_n$ are of the same nature.*
2. *If $\lim \frac{a_n}{b_n} = \ell \in (0, \infty)$, then the series $\sum a_n$ and $\sum b_n$ are of the same nature.*
3. *If $\lim \frac{a_n}{b_n} = 0$ and the series $\sum b_n$ is convergent, then the series $\sum a_n$ is convergent.*
4. *If $\lim \frac{a_n}{b_n} = \infty$ and the series $\sum b_n$ is divergent, then the series $\sum a_n$ is divergent.*

Compare the series $\sum \frac{n^2+1}{n^4+1}$, $\sum \frac{n^5+n}{n^6+1}$, $\sum \frac{n^{10}+1}{e^n+n^4}$ with the series $\sum \frac{1}{n^2}$, $\sum \frac{1}{n}$, $\sum e^{-\frac{n}{2}}$, respectively, to deduce their nature.

Comparision of series of positive terms

1. The first condition means that $mb_n \leq a_n \leq Mb_n$ for all $n \geq N$ and then $m \sum_{n=N}^{\infty} b_n \leq \sum_{n=N}^{\infty} a_n \leq M \sum_{n=N}^{\infty} b_n$. Because $\sum a_n$ and $\sum b_n$ are of positive terms, the two series are of the same nature.
2. The condition $\lim \frac{a_n}{b_n} = \ell > 0$ implies that there exists N such that for all $n \geq N$, $\frac{\ell}{2} \leq \frac{a_n}{b_n} \leq 2\ell$, and then they are of the same nature.
3. The condition $\lim \frac{a_n}{b_n} = 0$ implies that there exists N such that for all $n \geq N$, $\frac{a_n}{b_n} \leq 1$, that is $a_n \leq b_n$ and the result follows.
4. The condition $\lim \frac{a_n}{b_n} = \infty$ implies that there exists N such that for all $n \geq N$, $\frac{a_n}{b_n} \geq 1$, that is $a_n \geq b_n$ and the result follows.

Comparision of series of positive terms

Examples:

1. For the series $\sum \frac{n^2+1}{n^4+1}$, let $a_n = \frac{n^2+1}{n^4+1}$ and $b_n = \frac{1}{n^2}$. Both series $\sum a_n$ and $\sum b_n$ are of positive terms. Moreover, we have $\lim \frac{a_n}{b_n} = \lim \frac{n^2(n^2+1)}{n^4+1} = \lim \frac{n^4}{n^4} = 1$. But the series $\sum \frac{1}{n^2}$ is convergent. We deduce that the series $\sum \frac{n^2+1}{n^4+1}$ is convergent.
2. For the series $\sum \frac{n^5+n}{n^6+1}$, let $a_n = \frac{n^5+n}{n^6+1}$ and $b_n = \frac{1}{n}$. Both series $\sum a_n$ and $\sum b_n$ are of positive terms. Moreover, we have $\lim \frac{a_n}{b_n} = \lim \frac{n(n^5+n)}{n^6+1} = \lim \frac{n^6}{n^6} = 1$. But the series $\sum \frac{1}{n}$ is divergent. We deduce that the series $\sum \frac{n^5+n}{n^6+1}$ is divergent.
3. For the series $\sum \frac{n^{10}+1}{e^n+n^4}$, let $a_n = \frac{n^{10}+1}{e^n+n^4}$ and $b_n = \frac{1}{e^{\frac{n}{2}}}$. Both series $\sum a_n$ and $\sum b_n$ are of positive terms. Moreover, we have $\lim \frac{a_n}{b_n} = \lim \frac{e^{\frac{n}{2}}(n^{10}+1)}{e^n+n^4} = \lim \frac{n^{10}e^{\frac{n}{2}}}{e^n} = \lim \frac{n^{10}}{e^{\frac{n}{2}}} = 0$. But the series $\sum \frac{1}{e^{\frac{n}{2}}}$ is convergent. We deduce that the series

The ratio test

Theorem (The ratio test)

Let $\sum a_n$ be a series of positive terms for which $\lim \frac{a_{n+1}}{a_n}$ exists.

1. *If $\lim \frac{a_{n+1}}{a_n} < 1$, then the series $\sum a_n$ is convergent.*
2. *If $\lim \frac{a_{n+1}}{a_n} > 1$, then the series $\sum a_n$ is divergent.*
3. *If $\lim \frac{a_{n+1}}{a_n} = 1$, then the test fails.*

For the case $\lim \frac{a_{n+1}}{a_n} = 1$, consider the following examples:

1. The series $\sum \frac{1}{n}$ is of positive terms, satisfies this condition $\lim \frac{a_{n+1}}{a_n} = \lim \frac{n}{n+1} = 1$ and the series $\sum \frac{1}{n}$ is divergent.
2. The series $\sum \frac{1}{n^2}$ is of positive terms, satisfies this condition $\lim \frac{a_{n+1}}{a_n} = \lim \frac{n^2}{(n+1)^2} = 1$ and the series $\sum \frac{1}{n^2}$ is convergent.

This is to say that it is not possible to make any deduction from $\lim \frac{a_{n+1}}{a_n} = 1$.

The ratio test

For the other two cases:

1. If $\lim \frac{a_{n+1}}{a_n} = l < 1$, consider $l < r < 1$. There exists N such that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \leq r$. We have

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_n}{a_{n-1}} \leq r^{n-N}.$$

   Therefore, $a_n \leq \frac{a_N}{r^N} r^n$ for all $n \geq N$. But the series $\sum r^n$ is convergent and a_n is positive. We deduce that $\sum a_n$ is convergent.

2. If $\lim \frac{a_{n+1}}{a_n} = l > 1$, consider $l > r > 1$. There exists N such that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \geq r$. We find in the same way that $a_n \geq \frac{a_N}{r^N} r^n$ for all $n \geq N$. But the series $\sum r^n$ is divergent. We deduce that $\sum a_n$ is divergent.

The ratio test

Examples: Apply the ratio test to the series $\sum \frac{n!}{n^n}$ and $\sum \frac{n!^2}{(2n)!}$ to deduce their nature.

1. The series $\sum \frac{n!}{n^n}$ is of positive terms. Let $a_n = \frac{n!}{n^n}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

But we have already seen that $\lim \left(1+\frac{1}{n}\right)^n = e$. We deduce that $\lim \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$ and therefore, the series is convergent.

2. The series $\sum \frac{n!^2}{(2n)!}$ is of positive terms. Let $a_n = \frac{n!^2}{(2n)!}$. We have

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!^2(2n)!}{(2n+2)!(n!)^2} = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1.$$

We deduce that the series $\sum \frac{n!^2}{(2n)!}$ is convergent.

The root test

Theorem (The root test)

Let $\sum a_n$ be a series of positive terms for which $\lim \sqrt[n]{a_n}$ *exists.*

1. *If* $\lim \sqrt[n]{a_n} < 1$, *then the series* $\sum a_n$ *is convergent.*
2. *If* $\lim \sqrt[n]{a_n} > 1$, *then the series* $\sum a_n$ *is divergent.*
3. *If* $\lim \sqrt[n]{a_n} = 1$, *then the test fails.*

1. If $\lim \sqrt[n]{a_n} = l < 1$, consider $l < r < 1$. There exists N such that for all $n \geq N$, $\sqrt[n]{a_n} \leq r$ and then $a_n \leq r^n$. But the series $\sum r^n$ is convergent since $r < 1$ and a_n is positive. We deduce that $\sum a_n$ is convergent.
2. If $\lim \sqrt[n]{a_n} = l > 1$, consider $l > r > 1$. There exists N such that for all $n \geq N$, $\sqrt[n]{a_n} \geq r$ and then $a_n \geq r^n$. But the series $\sum r^n$ is divergent since $r > 1$. We deduce that $\sum a_n$ is divergent.

The root test

Examples: Apply the root test to the series $\sum \left(\frac{2n+1}{3n+1}\right)^n$ and $\sum \left(\frac{n}{n+1}\right)^{n^2}$ to deduce their nature.

1. The series $\sum \left(\frac{2n+1}{3n+1}\right)^n$ is of positive terms. Let $a_n = \left(\frac{2n+1}{3n+1}\right)^n$. We have $\lim \sqrt[n]{a_n} = \lim \frac{2n+1}{3n+1} = \frac{2}{3} < 1$. We deduce that the series $\sum \left(\frac{2n+1}{3n+1}\right)^n$ is convergent.
2. The series $\sum \left(\frac{n}{n+1}\right)^{n^2}$ is of positive terms. Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. We have $\lim \sqrt[n]{a_n} = \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$. We deduce that the series $\sum \left(\frac{n}{n+1}\right)^{n^2}$ is convergent.

Alternating series

Definition

An **Alternating series** is a series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$, where $\{a_n\}$ is a sequence of positive terms.

Examples: $\sum(-3)^n$, $\sum\left(-\frac{1}{2}\right)^n$, $\sum\frac{(-1)^n}{n}$, $\sum\frac{(-1)^n}{\sqrt{n}}$, $\sum\frac{(-1)^n}{n^2}$ are all alternating series.

Theorem

Let $\sum(-1)^n a_n$ be an alternating series. If the sequence $\{a_n\}$ is decreasing and $\lim a_n = 0$, then the series $\sum(-1)^n a_n$ is convergent.

Examples: All the sequences $\left\{\frac{1}{2^n}\right\}$, $\left\{\frac{1}{n}\right\}$, $\left\{\frac{1}{\sqrt{n}}\right\}$, $\left\{\frac{1}{n^2}\right\}$ are decreasing and have limit 0. Therefore, the alternating series $\sum\left(-\frac{1}{2}\right)^n$, $\sum\frac{(-1)^n}{n}$, $\sum\frac{(-1)^n}{\sqrt{n}}$, $\sum\frac{(-1)^n}{n^2}$ are all convergent.

Alternating series

Proof of the theorem: Consider the alternating series $\sum(-1)^n a_n$ and let $\{s_n\}$ be the sequence of its partial sums. We have $s_1 = -a_1$, $s_2 = -a_1 + a_2$, $s_3 = -a_1 + a_2 - a_3, \ldots$. Because the sequence $\{a_n\}$ is decreasing, we have:

- $s_{2(n+1)} - s_{2n} = a_{2n+2} - a_{2n+1} < 0$, for all $n \geq 1$, which means that the sequence $\{s_{2n}\}$ is decreasing.
- $s_{2(n+1)+1} - s_{2n+1} = -a_{2n+3} + a_{2n+2} > 0$ for all $n \geq 0$, which means that the sequence $\{s_{2n+1}\}$ is increasing.

Now, $s_2 \geq s_{2n} = s_{2n+1} + a_{2n+1} > s_{2n+1} \geq s_1$, for all $n \geq 1$. This means that the decreasing sequence $\{s_{2n}\}$ is bounded bellow by s_1 and the increasing sequence $\{s_{2n+1}\}$ is bounded above by s_2. Therefore, both sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent. Let s and s' be their respective limits. We have $s' - s = \lim s_{2n+1} - \lim s_{2n} = \lim(s_{2n+1} - s_{2n}) = \lim -a_{2n+1} = 0$. This proves that $s' = s$ and $\sum(-1)^n a_n$ is convergent.

Absolute convergence

In all what follow, the series $\sum a_n$ is not necessarily of positive terms. Some of the previous tests cannot be applied:

For example, the series $\sum \frac{(-1)^n}{\sqrt{n}}$ is convergent as an alternating series, while the series $\sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$ is divergent as a sum of a convergent and a divergent series. So, they are not of the same nature, although

$$\lim \frac{\frac{(-1)^n}{\sqrt{n}}}{\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}} = \lim \frac{1}{1 + \frac{(-1)^n}{\sqrt{n}}} = 1.$$

This means that the condition of positivity in the comparision test is important and cannot be dropped.

Absolute convergence

Definition

A series $\sum a_n$ is said to be **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Examples:

- The series $\sum \frac{(-1)^n}{n^2}$, $\sum \frac{(-1)^n}{n^3}$, $\sum \frac{(-1)^n}{2^n}$ are convergent and absolutely convergent.
- The series $\sum \frac{(-1)^n}{n}$, $\sum \frac{(-1)^n}{\sqrt{n}}$, $\sum \frac{(-1)^n \ln n}{n}$ are convergent but not absolutely convergent.
- The series $\sum (-1)^n$ is divergent and not absolutely convergent.

Theorem

Any absolutely convergent series is convergent.

Absolute convergence

Definition

We say that a series is **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem

If $\sum a_n$ is a conditionally convergent series and $l \in \mathbb{R}$ is any real number, then we can always rearrange the terms of the series so that the new series $\sum a_{\sigma(n)}$ converges to l.

Any series is of one of the following types:

- It is absolutely convergent;
- It is conditionally convergent;
- It is divergent.

The ratio test

If we drop the condition of positivity in the ratio test, the test still works and it becomes:

Theorem (The ratio test)

Let $\sum a_n$ be a series of non-zero terms for which $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists.

1. *If $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum a_n$ is absolutely convergent.*
2. *If $\lim \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum a_n$ is divergent.*
3. *If $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test fails.*

The second case comes from the fact that in the proof of the theorem, there exists $r > 1$ and $N \geq 1$ such that for all $n \geq N$, we have $|a_n| \geq Cr^n$, where $C > 0$. But $\lim r^n = \infty$. Therefore, $\lim |a_n| = \infty$. Hence $\lim a_n \neq 0$ and the series $\sum a_n$ is divergent.

The root test

In a similar way, if we drop the condition of positivity in the root test, the test still works and it becomes:

Theorem (The root test)

Let $\sum a_n$ *be a series for which* $\lim \sqrt[n]{|a_n|}$ *exists.*

1. *If* $\lim \sqrt[n]{|a_n|} < 1$, *then the series* $\sum a_n$ *is absolutely convergent.*
2. *If* $\lim \sqrt[n]{|a_n|} > 1$, *then the series* $\sum a_n$ *is divergent.*
3. *If* $\lim \sqrt[n]{|a_n|} = 1$, *then the test fails.*

The argument for the second case is similar to the one for the ratio test.

Exercises

Exercises: For each of the following series, say if it is absolutely convergent, conditionally convergent or divergent.

$$(i)\ \sum \frac{(-1)^n}{\ln(n+1)}, \quad (ii)\ \sum \frac{\cos\frac{2n\pi}{3}}{n}, \quad (iii)\ \sum \frac{(-n)^n}{n!},$$

$$(iv)\ \sum (-1)^n \left(\frac{\ln n}{n}\right)^{\ln n}, \quad (v)\ \sum \frac{\tan^{-1} n}{n^2+n+1},$$

$$(vi)\ \sum (-1)^n \sin\frac{20\pi}{n}.$$

Definition

A **power series** (centred at 0/in x) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

depending on the real variable x.

Examples of power series:

1. $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$
2. $$\sum_{n=1}^{\infty} \frac{x}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$
3. $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Problem: For which values of x, the power series is convergent? The set of such values is called the **domain of convergence** of the series. Notice that all power series are convergent at $x = 0$.

Examples:

1. The power series $\displaystyle\sum_{n=0}^{\infty} x^n$ is a geometric series. It is absolutely convergent when $|x| < 1$ and divergent when $|x| \geq 1$. Its domain of convergence is $(-1, 1)$.

2. For the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, the ratio test gives:

   $\lim \left| \dfrac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim \left| \dfrac{x}{n+1} \right| = 0$. The series is absolutely convergent for all $x \in \mathbb{R}$. Its domain of convergence is $\mathbb{R}$.

1. For the power series $\sum_{n=0}^{\infty} n!x^n$, the ratio test gives:
   $\lim \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim |(n+1)x| = 0$, for $x \neq 0$.
   The domain of convergence of the series is $\{0\}$.
2. For the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, the ratio test gives
   $\lim \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim \left|\frac{n}{n+1}x\right| = |x|$. The series is absolutely convergent for $|x| < 1$ and divergent for $|x| > 1$.
   - For $x = 1$, it is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.
   - For $x = -1$, it is the convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

   The domain of convergence of the series is $[-1, 1)$.

Radius of convergence

Theorem

If a power series is convergent at $x_0 \neq 0$, then it is absolutely convergent at all $|x| < |x_0|$.

This theorem means that the domain of convergence of any power series is an interval centred at 0. It can be open $(-r, r)$, closed $[-r, r]$ or half open $(-r, r]$ or $[-r, r)$, for some $r \in [0, \infty) \cup \{\infty\}$, called the **radius of convergence** of the series.

Examples:

1. The power series $\sum x^n$, $\sum nx^n$, $\sum \frac{x^n}{n}$ have radius of convergence $r = 1$.
2. The power series $\sum \frac{x^n}{n!}$ has radius of convergence $r = \infty$.
3. The power series $\sum n!x^n$ has radius of convergence $r = 0$.

Radius of convergence

Proof of the theorem:

Assume the power series $\sum_{n=0}^{\infty} a_n x^n$ convergent at $x_0 \neq 0$ and let $|x| < |x_0|$.

We have $\lim a_n x_0^n = 0$. Therefore, there exists N such that, for all $n \geq N$, $|a_n x_0^n| \leq 1$.

Hence, for all $n \geq N$, we have $|a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n \leq \left|\frac{x}{x_0}\right|^n$.

But, the geometric series $\sum \left|\frac{x}{x_0}\right|^n$ is convergent since $\left|\frac{x}{x_0}\right| < 1$.

We deduce that the series $\sum a_n x^n$ is absolutely convergent.