chapter 4 : ESTIMATION

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Q4.1 page70:

The data in Table 4.5 show the numbers of cases of AIDS in Australia by date of diagnosis for successive 3-months periods from 1984 to 1988.

(Data from National Centre for HIV Epidemiology and Clinical Research 1994.)

In this early phase of the epidemic, the numbers of cases seemed to be increasing exponentially.

a) Plot the number of cases y_i against time period i ($i = 1, \dots, 20$).

b) A possible model is the Poisson distribution with parameter $\boldsymbol{\lambda_i = i^{\theta}}$,

or equivalently

$$\log \lambda_i = \theta \log i$$

Plot $log\ y_i$ against $\log i$ to examine this model.

c) Fit a generalized linear model to these data using the Poisson distribution, the log-link function and the equation

$$g(\lambda_i) = \log \lambda_i = \beta_1 + \beta_2 x_i \,,$$

where $\boldsymbol{x_i = \log i}$. Firstly, do this from first principles, working out expressions for the weight matrix $\mathbf{W}$ and other terms needed for the iterative equation

$$\mathbf{X^T\, W\, X\, b^{(m)} = X^T\, W\, z}$$

and using software which can perform matrix operations to carry out the calculations.

d) Fit the model described in (c) using statistical software which can perform Poisson regression. Compare the results with those obtained in (c).

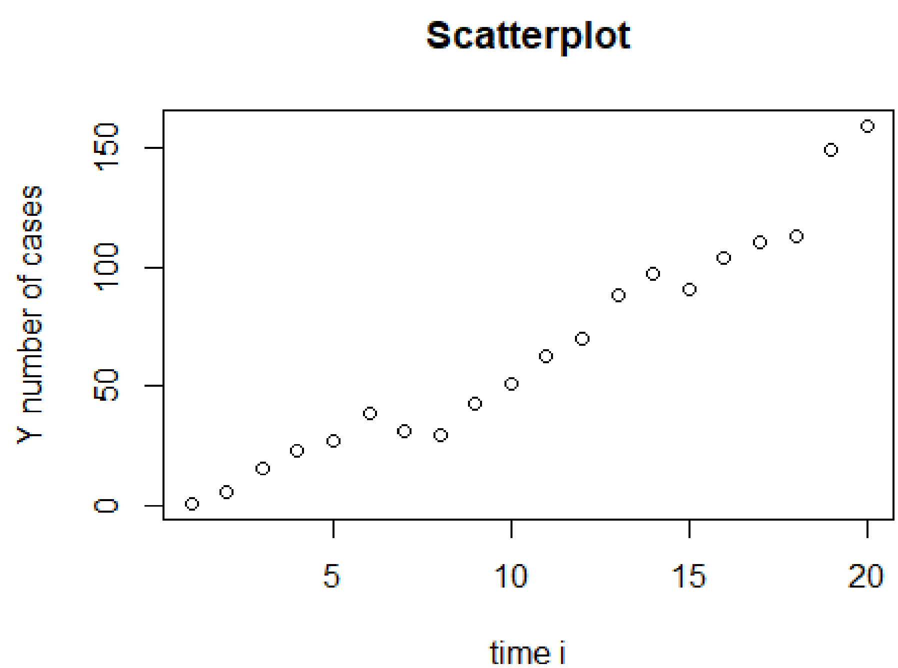
Table 4.5 *Numbers of cases of AIDS in Australia for successive quarter from 1984 to 1988.*

	Quarter			
Year	1	2	3	4
1984	1	6	16	23
1985	27	39	31	30
1986	43	51	63	70
1987	88	97	91	104
1988	110	113	149	159

```r
# y=numbers of cases of AIDS in Australia .
y<-c(1,6,16,23,27,39,31,30,43,51,63,70,88,97,91,104,110,113,149,159)
y

##  [1]   1   6  16  23  27  39  31  30  43  51  63  70  88  97  91 104 110 113 149
## [20] 159

#time period i (i = 1, . . . , 20).
i<-c(1:20)
i

##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
```

a) Plot the number of cases y_i against time period i ($i = 1,\ldots,20$).

```r
#   Plot the number of cases yi against time period i ( i = 1,...,20).
plot(i,y,main = "Scatterplot", xlab = "time i", ylab = "Y number of cases")
```



b) A possible model is the Poisson distribution with parameter $\lambda_i = i^{\theta}$,

or equivalently

$$\log \lambda_i = \theta \log i$$

Plot $log\ y_i$ against $\log i$ to examine this model.

$$Y_i \sim Poisson\ (\lambda_i)$$

Mean of Y_i: $E(Y_i) = \mu = \lambda_i$

Variance of Y_i: $Var(Y_i) = \sigma^2 = \lambda_i$

$\left.\begin{matrix} E(Y_i) = \mu = \lambda_i \\ Var(Y_i) = \sigma^2 = \lambda_i \end{matrix}\right\}$ Mean =Variance

$$\boldsymbol{\lambda_i = i^\theta} \gg \boldsymbol{\log \lambda_i = \theta \log(i)}$$

$$\boldsymbol{\log \lambda_i = \theta\, x_i}$$

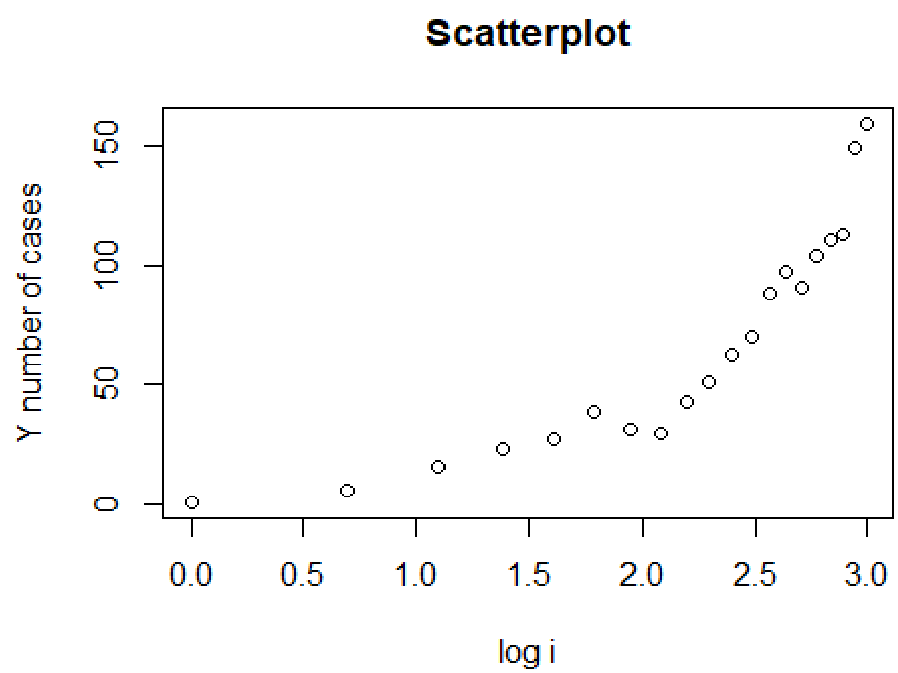
$$\boldsymbol{\log \lambda_i = \eta_i}$$

Where [$x_i = \log(i)$]
$\eta_i = \eta_i(x) = \theta\, x_i$

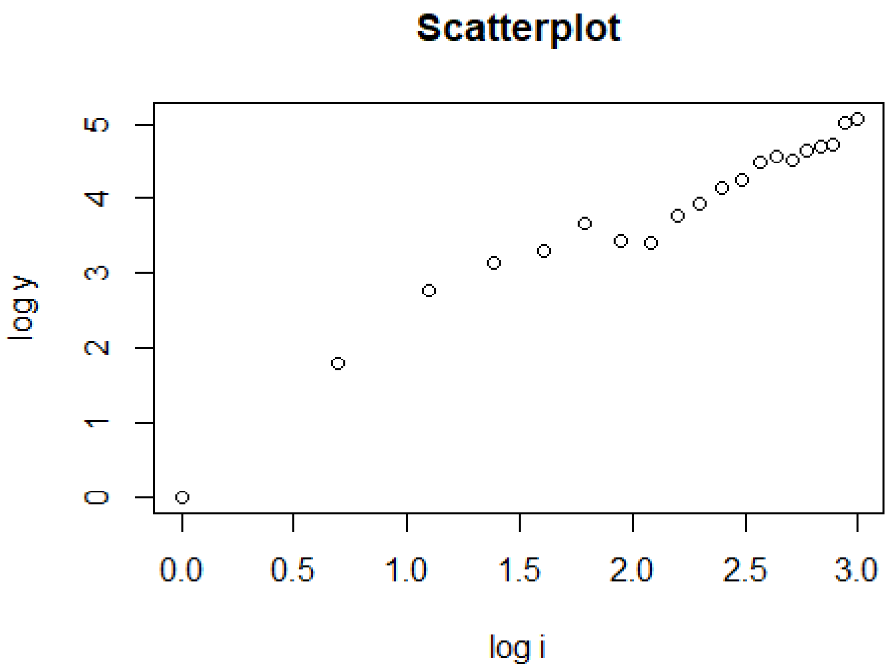
Link Function:

$$g(\boldsymbol{\lambda_i}) = \boldsymbol{\log \lambda_i}$$

```
#Plot  log yi against log(i)  to examine this model.
plot(log(i),y, main = "Scatterplot", xlab = "log i", ylab = "Y number of cases")
```




```
plot(log(i),log(y), main = "Scatterplot", xlab = "log i", ylab = "log y")
```



e) Fit a generalized linear model to these data using the Poisson distribution, the log-link function and the equation

$$g(\lambda_i) = \log \lambda_i = \beta_1 + \beta_2 x_i$$

$$\log \lambda_i = \beta_1 + \beta_2 x_i \gg \lambda_i = e^{\beta_1+\beta_2 x_i} \gg \lambda_i = e^{\eta_i}$$

where $x_i = \log(i)$:

$$\lambda_i = e^{\beta_1+\beta_2 x_i} \gg \lambda_i = e^{\beta_1+\beta_2 \log(\mathrm{i})}$$

$$\lambda_i = e^{\beta_1} . e^{\beta_2 \log(\mathrm{i})}$$

$$\lambda_i = e^{\beta_1} . e^{\log(\mathrm{i})^{\beta_2}}$$

$$\boldsymbol{\lambda_i = e^{\beta_1} . i^{\beta_2}}$$

We will use the iterative formula which is :

$$\mathbf{b}^{\mathbf{m+1}} = \mathbf{b}^{\mathbf{m}} + [\,\boldsymbol{\tau}(\mathbf{b}^{\mathbf{m}})]^{(-1)}\ \mathbf{U}(\mathbf{b}^{\mathbf{m}}) \quad , \mathbf{m} = \mathbf{0}, \mathbf{1}, \mathbf{2} \ldots \ldots$$

We will start with an initial value $b^{(0)} = \begin{bmatrix} b_1^{(0)} \\ b_2^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

```
# Initial values b0:
beta<-c(1,1)
beta

## [1] 1 1
```

$$\boxed{\begin{bmatrix} \boldsymbol{b_1^{(0)}} \\ \boldsymbol{b_2^{(0)}} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}}$$

Design matrix (X) :

$$X = \begin{bmatrix} 1 & \ln(i) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}$$

```
#Design matrix (X) :
X=matrix(c(rep(1,20),log(i)),nrow=20,ncol = 2,byrow = F)
X

##       [,1]      [,2]
##  [1,]    1 0.0000000
##  [2,]    1 0.6931472
##  [3,]    1 1.0986123
##  [4,]    1 1.3862944
##  [5,]    1 1.6094379
##  [6,]    1 1.7917595
##  [7,]    1 1.9459101
##  [8,]    1 2.0794415
##  [9,]    1 2.1972246
## [10,]    1 2.3025851
## [11,]    1 2.3978953
## [12,]    1 2.4849066
## [13,]    1 2.5649494
## [14,]    1 2.6390573
## [15,]    1 2.7080502
## [16,]    1 2.7725887
## [17,]    1 2.8332133
## [18,]    1 2.8903718
## [19,]    1 2.9444390
## [20,]    1 2.9957323
```

Transpose of design matrix (X^T) :

$$X^t = \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & \cdots & \ln(N) \end{bmatrix}$$

```
#Transpose of design matrix :
Xt<-t(X)
Xt

##      [,1]      [,2]     [,3]     [,4]     [,5]     [,6]    [,7]     [,8]
## [1,]    1 1.0000000 1.000000 1.000000 1.000000 1.000000 1.00000 1.000000
## [2,]    0 0.6931472 1.098612 1.386294 1.609438 1.791759 1.94591 2.079442
##          [,9]    [,10]    [,11]    [,12]    [,13]    [,14]   [,15]    [,16]
## [1,] 1.000000 1.000000 1.000000 1.000000 1.000000 1.000000 1.00000 1.000000
## [2,] 2.197225 2.302585 2.397895 2.484907 2.564949 2.639057 2.70805 2.772589
##         [,17]    [,18]    [,19]    [,20]
## [1,] 1.000000 1.000000 1.000000 1.000000
## [2,] 2.833213 2.890372 2.944439 2.995732
```

The matrix of working weights $W = diag\left[\frac{1}{var(y_i)}\left(\frac{\partial \mu}{\partial \eta}\right)^2\right]$ is :

$$W = diag\left[e^{\beta_1}\, i^{\beta_2}\right]_{NxN}$$

```
#The matrix of working weights (W) is :
W<-exp(beta[1])*diag(c(i^beta[2]),nrow = 20,ncol = 20)
W
```

```
##          [,1]     [,2]     [,3]     [,4]     [,5]     [,6]     [,7]     [,8]
##  [1,] 2.718282 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [2,] 0.000000 5.436564 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [3,] 0.000000 0.000000 8.154845  0.00000  0.00000  0.00000  0.00000  0.00000
##  [4,] 0.000000 0.000000 0.000000 10.87313  0.00000  0.00000  0.00000  0.00000
##  [5,] 0.000000 0.000000 0.000000  0.00000 13.59141  0.00000  0.00000  0.00000
##  [6,] 0.000000 0.000000 0.000000  0.00000  0.00000 16.30969  0.00000  0.00000
##  [7,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000 19.02797  0.00000
##  [8,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000 21.74625
##  [9,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [10,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [11,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [12,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [13,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [14,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [15,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [16,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [17,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [18,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [19,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
## [20,] 0.000000 0.000000 0.000000  0.00000  0.00000  0.00000  0.00000  0.00000
##          [,9]    [,10]   [,11]    [,12]    [,13]    [,14]    [,15]    [,16]
##  [1,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [2,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [3,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [4,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [5,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [6,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [7,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [8,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##  [9,] 24.46454  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
## [10,]  0.00000 27.18282  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
## [11,]  0.00000  0.00000 29.9011  0.00000  0.00000  0.00000  0.00000  0.00000
## [12,]  0.00000  0.00000  0.0000 32.61938  0.00000  0.00000  0.00000  0.00000
## [13,]  0.00000  0.00000  0.0000  0.00000 35.33766  0.00000  0.00000  0.00000
## [14,]  0.00000  0.00000  0.0000  0.00000  0.00000 38.05595  0.00000  0.00000
## [15,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000 40.77423  0.00000
## [16,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000 43.49251
## [17,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
## [18,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
## [19,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
## [20,]  0.00000  0.00000  0.0000  0.00000  0.00000  0.00000  0.00000  0.00000
##         [,17]    [,18]    [,19]    [,20]
##  [1,]  0.00000  0.00000  0.00000  0.00000
##  [2,]  0.00000  0.00000  0.00000  0.00000
##  [3,]  0.00000  0.00000  0.00000  0.00000
##  [4,]  0.00000  0.00000  0.00000  0.00000
##  [5,]  0.00000  0.00000  0.00000  0.00000
##  [6,]  0.00000  0.00000  0.00000  0.00000
##  [7,]  0.00000  0.00000  0.00000  0.00000
##  [8,]  0.00000  0.00000  0.00000  0.00000
```

```
##  [9,]  0.00000  0.00000  0.00000  0.00000
## [10,]  0.00000  0.00000  0.00000  0.00000
## [11,]  0.00000  0.00000  0.00000  0.00000
## [12,]  0.00000  0.00000  0.00000  0.00000
## [13,]  0.00000  0.00000  0.00000  0.00000
## [14,]  0.00000  0.00000  0.00000  0.00000
## [15,]  0.00000  0.00000  0.00000  0.00000
## [16,]  0.00000  0.00000  0.00000  0.00000
## [17,] 46.21079  0.00000  0.00000  0.00000
## [18,]  0.00000 48.92907  0.00000  0.00000
## [19,]  0.00000  0.00000 51.64735  0.00000
## [20,]  0.00000  0.00000  0.00000 54.36564
```

The information matrix $\tau = X^t * W * X$ is :

```
##Information matrix, Tau= Xt*W*X (Multiply the matrices):
tau<- Xt%*%W%*%X
tavu

##            [,1]     [,2]
## [1,]  570.8392 1439.608
## [2,] 1439.6080 3768.835
```

$$\boxed{\tau = X^t\, W\, X}$$

The score statistics are: (U1 and U2):

```
#The score statistics are: (U1 and U2
U1<-sum(y-exp(beta[1]+beta[2]*log(i)))
U2<-sum(y*log(i)-log(i)*exp(beta[1]+beta[2]*log(i)))
```

The vector of scores $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$

```
U<-matrix(c(U1,U2))
U

##            [,1]
## [1,]  740.1608
## [2,] 1956.7710
```

$$\boxed{U = \begin{bmatrix} U1 \\ U2 \end{bmatrix} = \begin{bmatrix} \sum(y_i - e^{\beta_1+\beta_2 \ln(i)}) \\ \sum(y_i \ln(i) - \ln(i)\, e^{\beta_1+\beta_2 \ln(i)}) \end{bmatrix}}$$

We make the following Calculation:

$$\mathbf{b}^{\mathbf{m}+1} = \mathbf{b}^{\mathbf{m}} + [\ \boldsymbol{\tau}(\mathbf{b}^{\mathbf{m}})]^{-1}\ \ \mathbf{U}(\mathbf{b}^{\mathbf{m}})$$

```
#We make the following Calculation:
b<-beta+solve(tau)%*%U
b

##          [,1]
## [1,] 0.652354
## [2,] 1.651991
```

$$\boxed{\mathbf{b}^1 = \mathbf{b}^0 + [\boldsymbol{\tau}^0]^{-1}\, \mathbf{U}^0}$$

Repeat the steps but change the value of initial value to the last value of beta you get it, and stop the iteration process if you get the same value of Beta (in two successive steps)

The following table summarizes the results of the iterative procedure:

m	**0**	**1**	**2**	**3**	**4**	**5**	**6**
b_1^m	1	0.652354	0.841856	0.98454	0.995952	0.995998	0.995998
b_2^m	1	1.651991	1.429568	1.33373	1.326639	1.32661	1.32661

Since $b^{(5)} = b^{(6)} = \begin{bmatrix} 0.995998 \\ 1.32661 \end{bmatrix}$, the iteration algorithm was terminated at step (5) .

The final approximation of MLE of $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ $is\ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.995998 \\ 1.32661 \end{bmatrix}$

OR By using Loop

```
# y=numbers of cases of AIDS in Australia .
y<-c(1,6,16,23,27,39,31,30,43,51,63,70,88,97,91,104,110,113,149,159)
y

##  [1]   1   6  16  23  27  39  31  30  43  51  63  70  88  97  91 104 110 113 149
## [20] 159

#time period i (i = 1, . . . , 20).
i<-c(1:20)
i

##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
```

```r
# Initial values
beta <- matrix(c(1, 1))
epsilon <- 1e-6
max_iter <- 100

############### By use Loop ###############

# Iterative process
for (iter in 1:max_iter) {
##Design matrix (X)
X=matrix(c(rep(1,20),log(i)),nrow=20,ncol = 2,byrow = F)
#Transpose of design matrix :
Xt<-t(X)
#Working weights matrix (W)= Diagonal matrix having 20 rows and 20 columns :
W<- exp(beta[1])*diag((i)^(beta[2]), nrow=20, ncol=20)
#Information matrix , Tau= Xt*W*X
# Multiply the matrices.
Tau<-Xt %*% W %*% X
# to calculate the inverse of Tau:
Tau_inver<-solve(Tau)
#The score statistics are: (U1 and U2)
U1=sum(y-exp(beta[1]+beta[2]*log(i)))
U2=sum(y*log(i)-log(i)*exp(beta[1]+beta[2]*log(i)))
#The vector of scores (U)
U<-matrix(c(U1,U2))
U
#iterative equation to find an approximate estimate of beta: b1,b2
b<- beta+(Tau_inver %*% U)
b
# Check for convergence
if (max(abs(b - beta)) < epsilon ) {
  break
}

# Update beta for the next iteration
beta <- b

# Print iteration results (optional)
cat("Iteration", iter, ": beta =", t(b), "\n")
}

## Iteration 1 : beta = 0.652354 1.651991
## Iteration 2 : beta = 0.8418567 1.429548
## Iteration 3 : beta = 0.9845403 1.33373
## Iteration 4 : beta = 0.9959522 1.326639
## Iteration 5 : beta = 0.995998 1.32661
```

d) Fit the model described in (c) using statistical software

```
model<-glm(y~log(i) ,family =poisson(link = "log"))
summary(model)

##
## Call:
## glm(formula = y ~ log(i), family = poisson(link = "log"))
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  0.99600    0.16971   5.869 4.39e-09 ***
## log(i)       1.32661    0.06463  20.525  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 677.264  on 19  degrees of freedom
## Residual deviance:  21.755  on 18  degrees of freedom
## AIC: 138.05
##
## Number of Fisher Scoring iterations: 4
```

$var(b_1) = 0.16971^2 = 0.02880148$

$var(b_2) = 0.06463^2 = 0.00417703$

- **Find the 95% confidence interval for β_1 and β_2 is:**

$$\mathbf{b_1 \pm 1.96\ se(b_1)} \gg \quad \mathbf{0.996 \pm 1.96 * 0.1697}$$

$$\mathbf{b_2 \pm 1.96\ se(b_2)} \gg \quad \mathbf{1.32661 \pm 1.96 * 0.06463}$$

```
CI_of_beta <- confint.default(model,level=0.95)
CI_of_beta

##                 2.5 %   97.5 %
## (Intercept) 0.6633773 1.328619
## log(i)      1.1999299 1.453289
```

Find approximately the variance-covariance matrix of the MLE b :

Final approximate of the inverse of the information matrix evaluated at b is:

For obtaining the estimated variance-covariance matrix of parameter estimate s in a fitted model:

The variance-covariance matrix of the **MLE** b ($\mathbf{cov(b) = \tau^{-1}}$) is

```
Tauinver<-vcov(model)
Tauinver

##             (Intercept)       log(i)
## (Intercept)  0.02880067 -0.010822607
## log(i)      -0.01082261  0.004177519
```

$$\tau^{-1} = cov(\hat{\beta}) = cov(b) = \begin{bmatrix} \mathrm{var}(b_1) & \mathrm{cov}(b_1, b_2) \\ \mathrm{cov}(b_1, b_2) & \mathrm{var}(b_2) \end{bmatrix}$$

$$cov(b) = \begin{bmatrix} 0.02880067 & -0.010822607 \\ -0.01082261 & 0.004177519 \end{bmatrix}$$

Find approximate of the information matrix evaluated at b

```
Tau_at_b<-solve(vcov(model))
Tau_at_b

##             (Intercept)     log(i)
## (Intercept)  1311.000      3396.379
## log(i)       3396.379      9038.301
```

$$\tau = cov(U) = \begin{bmatrix} \mathrm{var}(U_1) & \mathrm{cov}(U_1, U_2) \\ \mathrm{cov}(U_1, U_2) & \mathrm{var}(U_2) \end{bmatrix}$$

$$cov(U) = \begin{bmatrix} 1311.000 & 3396.379 \\ 3396.379 & 9038.301 \end{bmatrix}$$

Not:

We may use one of the following iterative equations to find an approximate estimate of $\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$:

$$\boldsymbol{b}^{(m+1)} = \boldsymbol{b}^{(m)} + [\boldsymbol{\mathcal{J}}^{(m)}]^{-1}\,\boldsymbol{U}^{(m)}$$

or

$$\boldsymbol{b}^{(m+1)} = \left[\boldsymbol{X}^t\,\boldsymbol{W}^{(m)}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^t\,\boldsymbol{W}^{(m)}\boldsymbol{z}^{(m)}$$