Numerical Methods

King Saud University

Aims

Chapter 2

Lecture #4

In this lecture, we will . . .

- Introduce the Secant Method
- Introduce the Multiplicity of a Root
- Introduce the Convergence of Iterative Methods
- Introduce the Systems of Nonlinear Equations

Secant Method

Since we known the main obstacle to using the Newton's method is that it may be difficult or impossible to differentiate the function $f(x)$. The calculation of $f'(x_n)$ may be avoided by approximating the slope of the tangent at $x = x_n$ by that of the chord joining the two points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$, see Figure 1.

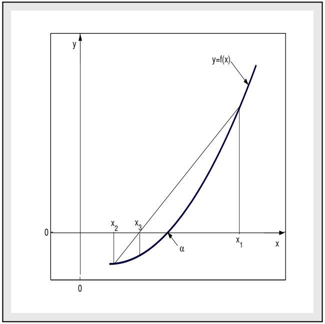


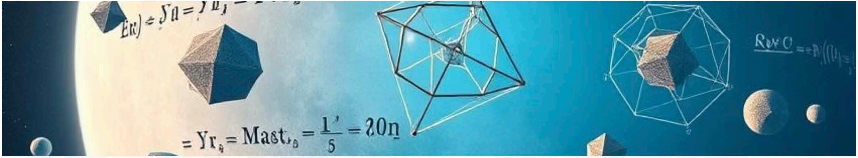
Figure 1: Graphical Solution of Secant Method.

The slope of the chord (or secant) is

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{1}$$

The iterative formula of the secant method is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1. \tag{2}$$



The Secant Method

1

Motivation

The Secant Method avoids the need to calculate derivatives, which can be difficult or impossible for some functions.

2

Approximation

It approximates the slope of the tangent at $x = x_n$ using the chord joining two points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$.

3

Formula

The iterative formula is:

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \qquad n \geq 1$$

Example 0.1

Use the secant method to find the approximate root of the following equation within the accuracy 10^{-2} take $x_0 = 1.5$ and $x_1 = 2.0$ as starting values

$$x^3 = 2x + 1.$$

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$$x^3 = 2x + 1.$$

Solution. Since $f(x) = x^3 - 2x - 1$ and

$$\begin{array}{lcll} x_0 & = & 1.5, & f(x_0) = -0.625, \\ x_1 & = & 2.0, & f(x_1) = 3.0, \end{array}$$

therefore, we see that $f(x_0) \neq f(x_1)$. Hence, one can use the iterative formula (2), to get new approximation:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.5)(3.0) - (2.0)(-0.625)}{3.0 - (-0.625)} = 1.586207,$$

and $f(x_2) = -0.18434$. Similar way, we can find the other possible approximation of the root. A summary of the calculations is given in Table 1. $\bullet$

Table 1: Solution of $x^3 = 2x + 1$ by secant method

n	x_{n-1}	x_n	x_{n+1}	$f(x_{n+1})$
01	1.500000	2.000000	1.586207	-0.1814342
02	2.000000	1.586207	1.609805	-0.0478446
03	1.586207	1.609805	1.618257	0.0013040

Example 0.2

Show that the iterative procedure for evaluating the reciprocal of a number N by using the secant method is:

$$x_{n+1} = x_n + (1 - Nx_n)x_{n-1}, \qquad n \geq 1. \tag{3}$$

Example 0.2

Show that the iterative procedure for evaluating the reciprocal of a number N by using the secant method is:

$$x_{n+1} = x_n + (1 - Nx_n)x_{n-1}, \qquad n \geq 1. \tag{3}$$

Solution. Let N be a positive number and $x = 1/N$. If $f(x) = 0$, then $x = \alpha = 1/N$ is the exact zero of the function

$$f(x) = 1/x - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1.$$

Hence, assuming the initial estimates to the root, say, $x = x_0, x = x_1$ and by using the secant iterative formula, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)(1/x_1 - N)}{(1/x_1 - N) - (1/x_0 - N)} = x_1 - \frac{(x_1 - x_0)(1/x_1 - N)}{-(x_1 - x_0)/x_1x_0}.$$

It gives

$$x_2 = x_1 + (1/x_1 - N)x_1x_0 = x_1 + x_0 - Nx_1x_0 = x_1 + (1 - Nx_1)x_0.$$

In general, this becomes

$$x_{n+1} = x_n + (1 - Nx_n)x_{n-1}, \qquad n = 1, 2, \ldots.$$

For example, suppose we want the reciprocal of number $N = 5$. Assuming the initial approximations of say $x_0 = 0$ and $x_1 = 0.1$, then by using the above iterative formula, we get the first three approximations as follows:

$$x_2 = 0.1, \qquad x_3 = 0.15, \qquad x_4 = 0.175,$$

The estimated value compares rather favorably with exact value of $1/5$, (see Figure 2). •

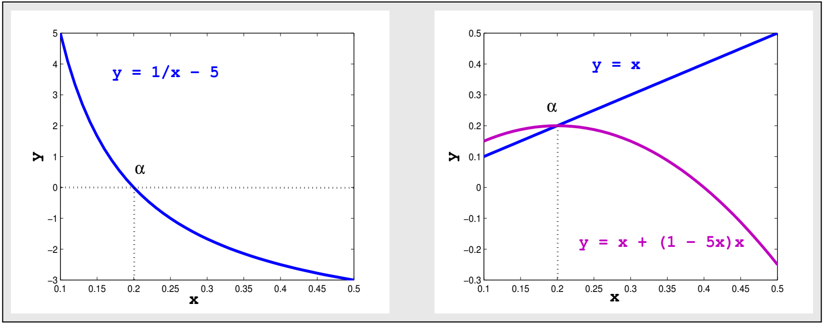


Figure 2: Graphical Solution of $1/x = 5$ and $x = x + (1 - 5x)x$.

Procedure

(Secant Method)

1. Choose the two initial approximation x_0 and x_1.
2. Check, if $f(x_0) = f(x_1)$, go to step *1* otherwise, continue.
3. Establish Tolerance ($\epsilon > 0$) value for the function.
4. Compute new approximation for the root by using the iterative formula (2).
5. Check tolerance. If $|x_n - x_{n-1}| \leq \epsilon$, for $n \geq 1$, then end; otherwise, go back to step *4*, and repeat the process.

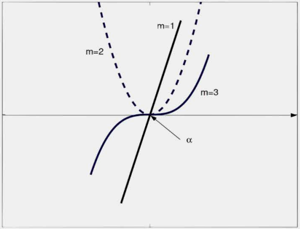
Multiplicity of a Root

1 **Definition**

A root α of order m exists if $f(x)$ can be expressed as $(x-\alpha)^m\, h(x)$, where $h(\alpha) \neq 0$.

2 **Simple vs Multiple Roots**

Simple roots are distinct, while multiple roots have the same order of magnitude.



3 **Graphical Behavior**

Odd multiplicity roots **cross the x-axis**, while **even** multiplicity roots are **tangent to it.**

4 **Identification**

Multiple roots satisfy

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0,\ f^{(m)}(\alpha) \neq 0$$

Lemma

Assume that function $f(x)$ and its derivatives $f'(x), f''(x), \cdots, f^{(m)}(x)$ are defined and continuous on an interval about $x = \alpha$. Then $f(x) = 0$ has a root α of *order* m if and only if

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0, \qquad f^{(m)}(\alpha) \neq 0. \tag{5}$$

For example, consider the equation $f(x) = x^3 - x^2 - 21x + 45 = 0$, which has three roots; a simple root at $\alpha = -5$ and a double root at $\alpha = 3$. This can be verified by considering the derivatives of the function as follows

$$f'(x) = 3x^2 - 2x - 21, \qquad f''(x) = 6x - 2.$$

At the value $\alpha = -5$, we have $f(5) = 0$ and $f'(5) = 64 \neq 0$, so by (5), we see that $m = 1$. Hence $\alpha = -5$ is a simple root of the equation. For the value $\alpha = 3$, we have

$$f(3) = 0, \qquad f'(3) = 0, \qquad f''(3) = 16 \neq 0,$$

so that $m = 2$ by (5), hence $\alpha = 3$ is a double root of the equation. Note that this function $f(x)$ has the factorization and can be written in the form of (4) as (see Figure 4),

$$f(x) = (x-3)^2(x+5).$$

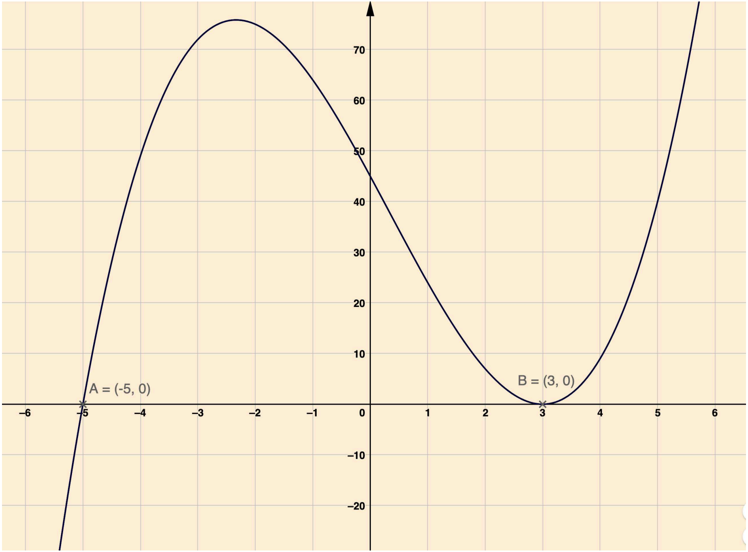
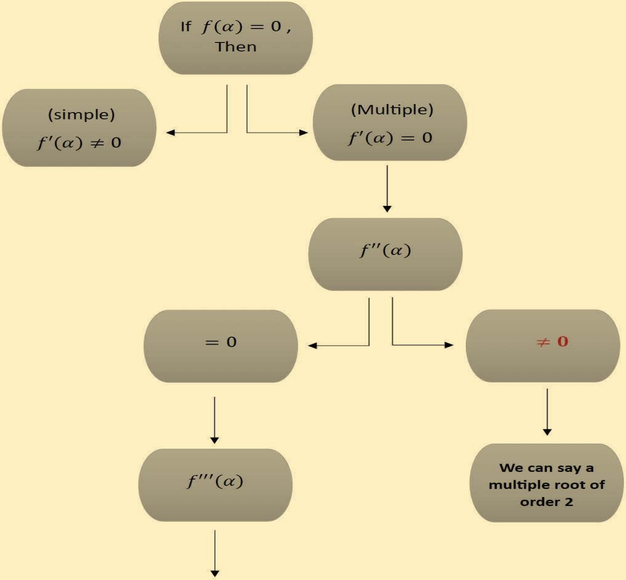


Figure 4: Graphical Solution of $x^3 - x^2 - 21x + 45 = 0$.

Multiple root:

- **How to find the order of multiple root:**



repeat to get the third approximation and so on.....

The order of multiplicity of the multiple root can be easily find out by taking the higher derivatives of the function at α unless the higher derivative becomes nonzero at α. Then the order of nonzero higher derivative will be the order of multiplicity of the multiple root. •

Example 0.3

Find the multiplicity of the root $\alpha = 1$ of the equation $x \ln x = \ln x$.

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Example 0.3

Find the multiplicity of the root $\alpha = 1$ of the equation $x \ln x = \ln x$.
Solution. From the given equation, we have

$$\begin{array}{lcllll} f(x) & = & x\ln x - \ln x & \text{and} & f(1) = & 0, \\ f'(x) & = & \ln x + 1 - \dfrac{1}{x} & \text{and} & f'(1) = & 0, \\ f''(x) & = & \dfrac{1}{x} + \dfrac{1}{x^2} & \text{and} & f''(1) = & 2 \neq 0. \end{array}$$

Thus the multiplicity of the root $\alpha = 1$ of the given equation is 2. •

Convergence of Iterative Methods

Usually we don't know in advance that an equation has multiple roots, although we might suspect it from sketching the graph. Many problems which leads to multiple roots, are in fact ill-posed. The methods we discussed so far cannot be guaranteed to converge efficiently for all problems. In particular, when a given function has a multiple root which we require, the methods we have described will either not converge at all or converge more slowly. For example, the Newton's method converges very fast to simple root but converges more slowly when used for functions involving multiple roots.

Example 0.4

Consider the following two nonlinear equations

$$(1) \qquad xe^x = 0 \qquad\qquad (2) \qquad x^2 e^x = 0.$$

(a) Find the Newton's method for the solutions of the given equations.
(b) Explain why one of the sequences converges much faster than the other to the root $\alpha = 0$.

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(a) Find the Newton's method for the solutions of the given equations.
(b) Explain why one of the sequences converges much faster than the other to the root $\alpha = 0$.
Solution. *(a)* For the first equation, we have

$$f(x) = xe^x \qquad \text{and} \qquad f'(x) = (1+x)e^x.$$

Then the Newton's method for the solution of the first equation is

$$x_{n+1} = g_1(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n^2}{(1+x_n)}, \qquad n \geq 0,$$

which is the first sequence. Similarly, we can find the Newton's method for the solution of the second equation as follows:

$$x_{n+1} = g_2(x_n) = x_n - \frac{x_n^2 e^{x_n}}{(2x_n + x_n^2)e^{x_n}} = \frac{x_n + x_n^2}{(2+x_n)}, \qquad n \geq 0,$$

and it is the second sequence.

(b) From the first sequence, we have

$$g_1(x) = \frac{x^2}{(1+x)} \quad \text{and} \quad g_1'(x) = \frac{x^2+2x}{(1+x)^2}.$$

Then

$$|g_1'(\alpha)| = |g_1'(0)| = \left|\frac{0}{1}\right| = 0,$$

which shows that the first sequence converges to zero. Similarly, from the second sequence, we have

$$g_2(x) = \frac{x+x^2}{(2+x)} \quad \text{and} \quad g_2'(x) = \frac{x^2+4x+2}{(2+x)^2}.$$

Thus

$$|g_2'(0)| = \left|\frac{2}{4}\right| = \frac{1}{2} < 1,$$

which shows that the second sequence is also converges to zero. Since the value of $|g_1'(0)|$ is smaller than $|g_2'(0)|$, therefore, the first sequence converges faster than the second one. •

Note that in the above Example 0.4, the root $\alpha = 0$ is the simple root for the first equation (see Figure 5) because

$$f(0) = 0 \qquad \text{but} \qquad f'(0) = 1 \neq 0,$$

and for the second equation it is a multiple root because

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 0.$$

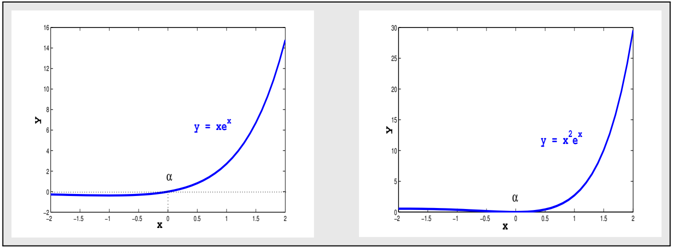

Figure 5: Graphical Solution of $xe^x = 0$ and $x^2e^x = 0$.

Therefore, the Newton's method converges very fast for the first equation and converges very slow for the second equation. However, in some cases simple modifications can be made to the methods to maintain the rate of convergence. Two such modified methods are considered here, called the Newton modified methods.

First Modified Newton's Method

To determine a root of known multiplicity m for the equation $f(x) = 0$, we may use the *first Newton's modified method* (also called the *Schroeder's method*) which is given by the form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots \tag{6}$$

Example 0.5

Show that nonlinear equation $\frac{1}{e^{(1-x)}} = x$ has a root at $x = 1$. Use the first modified Newton's method to find its first three approximations using $x_0 = 0$.

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Solution. Since $f(x) = 1 - xe^{(1-x)}$. First we show that $\alpha = 1$ is the zero of the given function as

$$f(\alpha) = f(1) = 1 - 1e^0 = 1 - 1 = 0.$$

To check whether it is *simple* or *multiple* zero of $f(x)$, we do the following

$$f'(x) = -e^{1-x} + xe^{(1-x)} \qquad \text{and} \qquad f'(\alpha) = f'(1) = -1 + 1 = 0,$$

which means that $\alpha = 1$ is the multiple zero of the given function. To find its order of multiplicity, we do

$$f''(x) = 2e^{(1-x)} - xe^{(1-x)} \qquad \text{and} \qquad f''(\alpha) = f''(1) = 2 - 1 = 1 \neq 0,$$

hence $\alpha = 1$ is a zero of multiplicity 2 of the given function. Now we have to find the first three approximations to the multiple zero $\alpha = 1$ of the given function by using the first modified Newton's method which can be written as

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - m\frac{1 - x_n e^{(1-x_n)}}{(x_n - 1)e^{(1-x_n)}}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function.

For $n = 0, 1, 2$ and $m = 2$, with initial approximation $x_0 = 0$, we have

$$x_1 = 0.7358, \qquad x_2 = 0.9782, \qquad x_3 = 0.9998,$$

are the required first three approximations to $\alpha = 1$, (see Figure 6). $\bullet$

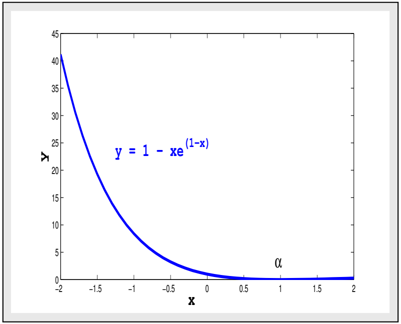

Figure 6: Graphical Solution of $1 - xe^{(1-x)} = 0$.

Second Modified Newton's Method

An alternative approach to this problem that does not require any knowledge of the multiplicity of the root is to replace the function $f(x)$ in the equation by $q(x)$, where

$$q(x) = \frac{f(x)}{f'(x)}.$$

This iterative formula of the second modified Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - [f(x_n)][f''(x_n)]}, \quad n = 0, 1, 2, \ldots \tag{7}$$

Example 0.6

Use the second modified Newton's method to find the first approximation x_1 to the multiple root of the nonlinear equation $1 - \cos(x) = 0$, using $x_0 = 0.1$.

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Example 0.6

Use the second modified Newton's method to find the first approximation x_1 to the multiple root of the nonlinear equation $1 - \cos(x) = 0$, using $x_0 = 0.1$.
Solution. Since $f(x) = 1 - \cos x$, we have $f'(x) = \sin x$ and $f''(x) = \cos x$. Now using the second modified Newton's formula (7)

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - [f(x_n)][f''(x_n)]}, \qquad n \geq 0,$$

we have

$$x_{n+1} = x_n - \frac{(1 - \cos x_n)(\sin x_n)}{[\sin x_n]^2 - (1 - \cos x_n)(\cos x_n)}, \qquad n \geq 0.$$

For $n = 0$ and the initial approximation $x_0 = 0.1$, we have

$$x_1 = x_0 - \frac{(1 - \cos x_0)(\sin x_0)}{[\sin x_0]^2 - (1 - \cos x_0)(\cos x_0)} = 0.1 - \frac{(1 - \cos 0.1)(\sin 0.1)}{[\sin 0.1]^2 - (1 - \cos 0.1)(\cos 0.1)} = 0.098,$$

which is the required first approximation to $\alpha = 0$, (see Figure 7). •

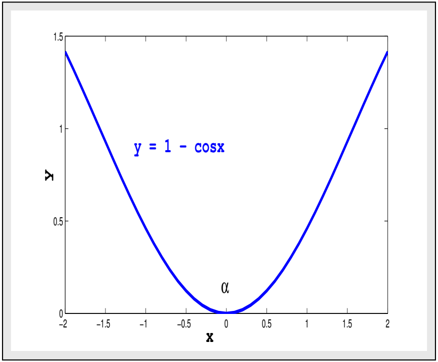

Figure 7: Graphical Solution of $1 - cosx = 0$.

Example 0.7

Show that the function $f(x) = e^x - \frac{x^2}{2} - x - 1$ has zero of multiplicity 3 at $\alpha = 0$ and then, find the approximate solution of the zero of the function with the help of the Newton's method, first and second modified Newton's methods, by taking initial approximation $x_0 = 1.5$ within an accuracy of 10^{-4}.

Example 0.7

Show that the function $f(x) = e^x - \frac{x^2}{2} - x - 1$ has zero of multiplicity 3 at $\alpha = 0$ and then, find the approximate solution of the zero of the function with the help of the Newton's method, first and second modified Newton's methods, by taking initial approximation $x_0 = 1.5$ within an accuracy of 10^{-4}.

Solution. Since $\alpha = 0$ is a root of $f(x)$, (see Figure 8), so

$$\begin{array}{lllllll} f(x) & = & e^x - \frac{x^2}{2} - x - 1, & f(0) & = & 0, \\ f'(x) & = & e^x - x - 1, & f'(0) & = & 0, \\ f''(x) & = & e^x - 1, & f''(0) & = & 0, \\ f'''(x) & = & e^x, & f'''(0) & = & 1 \neq 0, \end{array}$$

the function has zero of multiplicity 3.

Example 0.7

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Solution. Since $\alpha = 0$ is a root of $f(x)$, (see Figure 8), so

$$
\begin{array}{lclllcl}
f(x) & = & e^x - \frac{x^2}{2} - x - 1, & f(0) & = & 0, \\
f'(x) & = & e^x - x - 1, & f'(0) & = & 0, \\
f''(x) & = & e^x - 1, & f''(0) & = & 0, \\
f'''(x) & = & e^x, & f'''(0) & = & 1 \neq 0,
\end{array}
$$

the function has zero of multiplicity 3. In Table 2 we showed the comparison of three methods. •

We note that for the multiple root the both modified Newton's methods converge very fast as they took 4 iterations to converge while the Newton's method converges very slow and took 25 iterations to converge for the same accuracy.

Table 2: Comparison results of three methods for the Example 0.7

n	Newton's Method x_n	1st. M.N. Method x_n	2nd. M.N. Method x_n
00	1.500000	1.500000	1.500000
01	1.067698	0.2030926	-0.297704
02	0.745468	3.482923e-03	-6.757677e-03
03	0.513126	1.010951e-06	-3.798399e-06
..	....		
25	7.331582e-05		

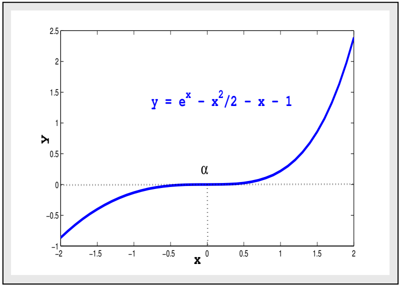


Figure 8: Graphical Solution of $e^x - x^2/2 - x - 1 = 0$.

Now we define the order of the convergence of functional iteration schemes discussed in the previous sections. This is a measure of how rapidly a sequence converges.

Definition 2

(Order of Convergence)
Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to α, and let $e_n = \alpha - x_n$ define the error of the *nth* iterate. If two positive constants $\beta \neq 0$ and $R > 0$ exist, and

$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^R} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^R} = \beta, \tag{8}$$

then the sequence is said to converge to α with *order of convergence* R. The number β is called the asymptotic error constant. The cases $R = 1, 2$ are given special consideration.

If $R = 1$, the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ is called ***linear.***

If $R = 2$, the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ is called ***quadratic.***

If R is large, the sequence $\{x_n\}$ converges rapidly to α; that is, (8) implies that for large values of n we have the approximation $|e_{n+1}| \approx \beta |e_n|^R$. For example, suppose that $R = 2$ and $|e_n| \approx 10^{-3}$; then we could expect that $|e_{n+1}| \approx \beta \times 10^{-6}$. •

Example 0.8

Show that the following sequence

$$x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{N}{x_n^2}\right), \qquad n \geq 0,$$

will converge quadratically to $\sqrt{N}$.

Example 0.8

Show that the following sequence

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right), \qquad n \geq 0,$$

will converge quadratically to $\sqrt{N}$.

Solution. Since the sequence is given as

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right),$$

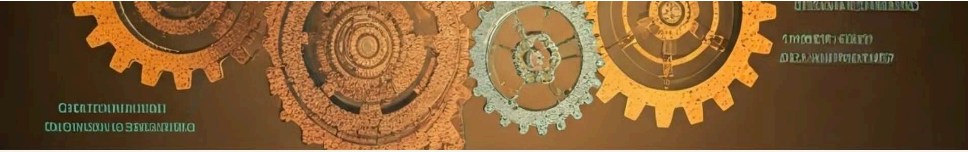
and $\alpha = \sqrt{N}$, then we have

$$\begin{aligned} x_{n+1} - \sqrt{N} &= \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right) - \sqrt{N} = \frac{1}{2}\left(x_n + \frac{N}{x_n} - 2\sqrt{N}\right) \\ &= \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{N}}{\sqrt{x_n}}\right)^2 = \frac{1}{2x_n}(x_n - \sqrt{N})^2. \end{aligned}$$

Thus

$$e_{n+1} = \frac{1}{2x_n}e_n^2 \qquad \text{or} \qquad e_{n+1} \propto e_n^2,$$

which shows the quadratic convergence. $\bullet$



Convergence of Iterative Methods

Rate of convergence:

1) linear: When $g'(a) \neq 0$
2) quadratic: when $g'(a) = 0 \ and \ g''(a) \neq 0$

   At least quadratic $g'(a) = 0$ only.

3) Cubic: when $g'(a) = 0, \ g''(a) = 0 \ and \ g'''(a) \neq 0$

   At least Cubic $g'(a) = 0, \ g''(a) = 0$ only.

Lemma 2.3 (Linear Convergence)

Let g is continuously differentiable on the interval $[a, b]$ and suppose that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that $g'(x)$ is continuous on (a, b) with

$$|g'(x)| \leq k < 1; \qquad \text{for all} \quad x \in (a, b).$$

If $g'(\alpha) \neq 0$, then for any $x_0 \in [a, b]$, the sequence $x_{n+1} = g(x_n)$, for $n \geq 0$, converges only linearly to the unique fixed-point α in $[a, b]$. •

Example 2.36 *Consider an iterative scheme*

$$x_{n+1} = 0.4 + x_n - 0.1x_n^2, \qquad n \geq 0.$$

Will this scheme converge to the fixed-point $\alpha = 2$? If yes, find its rate of convergence.

Solution. *Since*

$$g(x) = 0.4 + x - 0.1x^2 \quad \text{and} \quad g(2) = 0.4 + 2 - 0.1(2)^2 = 2,$$

which shows that the scheme converges to $\alpha = 2$. Also

$$g'(x) = 1 - 0.2x, \quad \text{gives}, \quad g'(2) = 1 - 0.4 = 0.6 \neq 0.$$

Therefore, the scheme converges linearly. •

Lemma 2.4 (Quadratic Convergence)

Let α be a solution of the equation $x = g(x)$. Suppose that $g'(\alpha) = 0$ and g'' is continuous on an open interval (a, b) containing α. Then there exists a $\delta > 0$ such that, for $x_0 \in [\alpha - \delta, \alpha + \delta]$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the iteration $x_{n+1} = g(x_n)$, for $n \geq 0$, converges at least quadratically to α. •

Example 2.37 *The iterative scheme*

$$x_{n+1} = 2 - (1 + a)x_n + ax_n^2, \qquad n \geq 0,$$

converges to $\alpha = 1$ for some values of a. Find the value of a for which the convergence is at least quadratic.

Example 2.37 *The iterative scheme*

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converges to $\alpha = 1$ for some values of a. Find the value of a for which the convergence is at least quadratic.

Solution. *Given*

$$g(x) = 2 - (1+a)x + ax^2 \quad \text{and} \quad g(1) = 2 - (1+a) + a = 1.$$

Thus, the given iterative scheme converges to 1. Also

$$g'(x) = -(1+a) + 2ax,$$

and so

$$g'(1) = 0 = -(1+a) + 2a, \quad \text{gives,} \quad a = 1.$$

Thus, the convergence of the given iterative scheme is at least quadratic for the value of $a = 1$. •

Note 2.3 *The sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by the iteration*

$$x_{n+1} = g(x_n), \quad \textit{for} \qquad n \geq 0,$$

converges only quadratically to α *if*

$$g'(\alpha) = 0 \quad \textit{but} \quad g''(\alpha) \neq 0.$$

and cubically (order three) to α *if*

$$g'(\alpha) = 0, \quad g''(\alpha) = 0 \quad \textit{but} \quad g'''(\alpha) \neq 0.$$

In the similar manner the higher order of convergence can be achieved.

Systems of Nonlinear Equations

1 — <u>Definition</u>

A system of n equations in n unknowns where at least one equation is nonlinear. This kind of system frequently arises in optimization problems, where we aim to find the minimum or maximum value of a function subject to constraints, and in numerical integration, where we approximate the value of a definite integral using numerical methods.

2 — <u>Two-Variable System</u>

Consider a system with two equations, $f_1(x, y) = 0$ and $f_2(x, y) = 0$, involving two variables, x and y . Our objective is to find values for x and y , denoted as α and β respectively, that simultaneously satisfy both equations.

3 — <u>Graphical Interpretation</u>

The solutions to this system correspond to the intersection points of the curves represented by the equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$ in the xy -plane. These intersection points represent the values of (x, y) that satisfy both equations simultaneously.

Newton's Method for Systems

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

We call the following matrix J a *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}$$

Example 0.9

For the following system of two equations

$$\begin{array}{rcl} x^3 + 3y^2 & = & 21 \\ x^2 + 2y & = & -2 \end{array}$$

Find the Jacobian matrix and its inverse using initial approximation $(1, -1)$, then find the first approximation by using the Newton's method.

Example 0.9

For the following system of two equations

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Find the Jacobian matrix and its inverse using initial approximation $(1, -1)$, then find the first approximation by using the Newton's method.

Solution. Given

$$\begin{array}{rclll} f_1(x, y) & = & x^3 + 3y^2 - 21, & {f_1}_x = 3x^2, & {f_1}_y = 6y, \\ f_2(x, y) & = & x^2 + 2y + 2, & {f_2}_x = 2x, & {f_2}_y = 2. \end{array}$$

At the given initial approximation $x_0 = 1$ and $y_0 = -1$, we have

$$\begin{array}{rclrclrcl} f_1(1, -1) & = & -17, & \dfrac{\partial f_1}{\partial x} = {f_1}_x & = & 3, & \dfrac{\partial f_1}{\partial y} = {f_1}_y & = & -6, \\ f_2(1, -1) & = & 1, & \dfrac{\partial f_1}{\partial x} = {f_2}_x & = & 2, & \dfrac{\partial f_2}{\partial y} = {f_2}_y & = & 2. \end{array}$$

The Jacobian matrix J at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{18} \begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix},$$

is the inverse of the Jacobian matrix. Now to find the first approximation we have to solve the following equation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{18} \begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -17 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5556 \\ -3.0556 \end{pmatrix},$$

the required first approximation. •

Example 0.10

Solve the following system of two equations using the Newton's method with given accuracy $\epsilon = 10^{-5}$.

$$\begin{aligned} 4x^3 + y &= 6 \\ x^2 y &= 1 \end{aligned}$$

Assume $x_0 = 1.0$ and $y_0 = 0.5$ as starting values.

Example 0.10

Solve the following system of two equations using the Newton's method with given accuracy $\epsilon = 10^{-5}$.

$$\begin{array}{rcl} 4x^3 + y & = & 6 \\ x^2 y & = & 1 \end{array}$$

Assume $x_0 = 1.0$ and $y_0 = 0.5$ as starting values.

Solution. Obviously this system of nonlinear equations has an exact solution of $x = 1.088282$ and $y = 0.844340$, (see Figure 10). Let us look how the Newton's method is used to approximate these roots. The first partial derivatives are as follows:

$$\begin{array}{rcl} f_1(x,y) & = & 4x^3 + y - 6, \quad f_{1x} = 12x^2, \quad f_{1y} = 1, \\ f_2(x,y) & = & x^2 y - 1, \quad f_{2x} = 2xy, \quad f_{2y} = x^2. \end{array}$$

At the given initial approximation $x_0 = 1.0$ and $y_0 = 0.5$, we get

$$\begin{array}{rcl} f_1(1.0, 0.5) & = & -1.5, \quad \dfrac{\partial f_1}{\partial x} = f_{1x} = 12, \quad \dfrac{\partial f_1}{\partial y} = f_{1y} = 1.0, \\ f_2(1.0, 0.5) & = & -0.5, \quad \dfrac{\partial f_1}{\partial x} = f_{2x} = 1.0, \quad \dfrac{\partial f_2}{\partial y} = f_{2y} = 1.0. \end{array}$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as follows:

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 12.0 & 1.0 \\ \\ 1.0 & 1.0 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{11.0} \begin{pmatrix} 1.0 & -1.0 \\ -1.0 & 12.0 \end{pmatrix}.$$

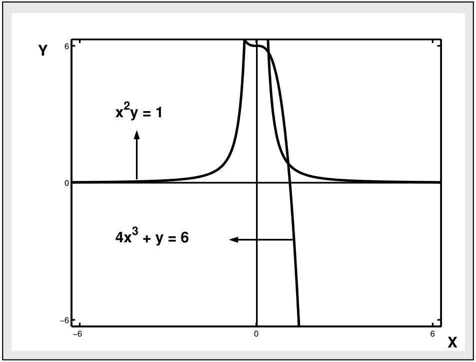

Figure 10: Graphical solution of the given nonlinear system.

By using the following formula

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{18}$$

we get the first approximation as follows

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} - \frac{1}{11.0} \begin{pmatrix} 1.0 & -1.0 \\ -1.0 & 12.0 \end{pmatrix} \begin{pmatrix} -1.5 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 1.090909 \\ 0.909091 \end{pmatrix}.$$

Similarly, the second iteration gives

$$\begin{aligned} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1.090909 \\ 0.909091 \end{pmatrix} - \frac{1}{15.012077} \begin{pmatrix} 1.190082 & -1.0 \\ -1.983471 & 14.280989 \end{pmatrix} \begin{pmatrix} 0.102178 \\ 0.081893 \end{pmatrix} \\ &= \begin{pmatrix} 1.088264 \\ 0.844686 \end{pmatrix}. \end{aligned}$$

The first two and the further steps of the method are listed in Table 3. •

Table 3: Solution of a system of two nonlinear equations

n	x-approx. x_n	y-approx. y_n	1st. func. $f_1(x_n, y_n)$	2nd. func. $f_2(x_n, y_n)$
00	1.000000	0.500000	-1.50000	-0.500000
01	1.090909	0.909091	0.102178	0.081893
02	1.088264	0.844686	0.000091	0.000377
03	1.088282	0.844340	0.000001	0.000001

1. Choose the initial guess for the roots of the system, so that the determinant of the Jacobian matrix is not zero.
2. Establish Tolerance $\epsilon (> 0)$.
3. Evaluate the Jacobian at initial approximations and then find inverse of Jacobian.
4. Compute new approximation to the roots by using iterative formula
$$\mathbf{x}^{[n+1]} = \mathbf{Z}^{[n]} + \mathbf{x}^{[n]}. \tag{19}$$
.
5. Check tolerance limit. If $\|(x_n, y_n) - (x_{n-1}, y_{n-1})\| \leq \epsilon$, for $n \geq 0$, then end; otherwise, go back to step *3*, and repeat the process.

Summary

In this lecture, we ...

- Introduced the Secant Method
- Introduced the Multiplicity of a Root
- Introduced the Convergence of Iterative Methods
- Introduced the Systems of Nonlinear Equations