Numerical Methods

King Saud University

Chapter 2

Lecture #3

In this lecture, we will . . .

- Introduce the Newton's Method

Newton's Method

This is one of the most popular and powerful iterative method for finding roots of the nonlinear equation. It is also known as the method of tangents because after estimated the actual root, the zero of the tangent to the function at that point is determined. The Newton's method consists geometrically of expanding the tangent line at a current point x_i until it crosses zero, then setting the next guess x_{i+1} to the abscissa of that zero crossing, see Figure 5. This method is also called the *Newton-Raphson method*.

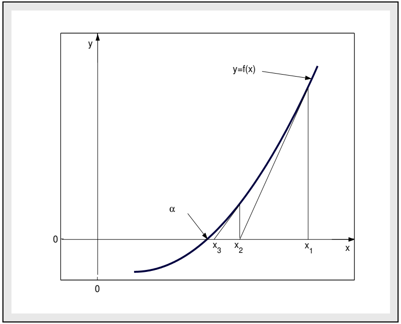

Figure 5: Graphical Solution of Newton's Method.

There are many description of the Newton's method. We shall derive the method from the familiar Taylor's series expansion of a function in the neighborhood of a point. Let $f \in C^2[a,b]$ and let x_n be the *nth* approximation to the root α such that $f'(x_n) \neq 0$ and $|\alpha - x_n|$ is small. Consider the first Taylor polynomial for $f(x)$ expanded about x_n, so we have

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x - x_n)^2}{2} f''(\eta(x)), \tag{7}$$

where $\eta(x)$ lies between x and x_n. Since $f(\alpha) = 0$, then (7), with $x = \alpha$, gives

$$f(\alpha) = 0 = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\eta(\alpha)).$$

Since $|\alpha - x_n|$ is small, then we neglect the term involving $(\alpha - x_n)^2$ and so

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n).$$

Solving for α, we get

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8}$$

which should be better approximation to α than is x_n. We call this approximation as x_{n+1}, then we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \qquad \text{for all } n \geq 0. \tag{9}$$

The iterative method (9) is called the Newton's method.

Example 0.1

Use Newton's method to find the approximation x_3 to the root of

$$\cos x - x = 0,$$

where $x_0 = \pi/4$. **Solution.**

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Let $f(x) = \cos x - x = 0$, so f '(x) =- sin x – 1 and use Using the Newton's iterative formula (9), we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to find the iterations, where $x_0 = \pi/4$. Thus we get:

Table: Solution of $\cos x - x = 0$ by Newton's method

n	x_n
1	0.7853981635
2	0.7395361337
3	0.7390851781

Therefore, $x_3 = 0.7390851781$.

Example 0.2

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Solution. Given $f(x) = x^3 - 2x - 1$ and so $f'(x) = 3x^2 - 2$. Now evaluating $f(x)$ and $f'(x)$ at the give approximation $x_0 = 1.5$, gives

$$x_0 = 1.5, \qquad f(1.5) = -0.625, \qquad f'(1.5) = 4.750.$$

Using the Newton's iterative formula (9), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.625)}{4.75} = 1.631579.$$

Now evaluating $f(x)$ and $f'(x)$ at the new approximation x_1, gives

$$x_1 = 1.631579, \qquad f(1.631579) = 0.0801869, \qquad f'(1.631579) = 5.9861501.$$

Using the iterative formula (9) again to get other new approximation. The successive iterates were shown in the Table 3.

Table: Solution of $x^3 = 2x + 1$ by Newton's method

n	x_n	$f(x_n)$	$f'(x_n)$	Error $= x - x_n$
00	1.500000	-0.625000	4.750000	0.1180339
01	1.631579	0.0801869	5.9861501	-0.0135451
02	1.618184	0.000878	5.855558	-0.0001501
03	1.618034	0.00000007	5.854102	-0.0000001

Just after the third iterations the required root is approximated to be $x_3 = 1.618034$ and the functional value is reduced to 7.0×10^{-8}. Since the exact solution is 1.6180339, so the actual error is 1×10^{-7}. We see that the convergence is quite faster than the methods considered previously (See Example 0.1 in Ch2_lecture1).

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Example 0.3

If the difference of two numbers x and y is 6 and the square root of their product is 4, then use Newton's method to approximate, to within 10^{-4}, Find the largest value of the number x and the corresponding number y using initial approximation $x_0 = 7.5$.

Example 0.3

If the difference of two numbers x and y is 6 and the square root of their product is 4, then use Newton's method to approximate, to within 10^{-4}, Find the largest value of the number x and the corresponding number y using initial approximation $x_0 = 7.5$.

Solution. Given

$$x - y = 6 \quad \text{and} \quad \sqrt{xy} = 4.$$

Solving the above equations for x, we have

$$x(x-6) = 16 \quad \text{or} \quad x^2 - 6x - 16 = f(x) = 0.$$

Applying Newton's iterative formula (9) to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 6x_n - 16}{2x_n - 6}.$$

Finding the approximation to within 10^{-4} using the initial approximation $x_0 = 7.5$, we get

$$x_1 = x_0 - \frac{x_0^2 - 6x_0 - 16}{2x_0 - 6} = 8.0278,$$

and continue in the same manner, we get the approximations within accuracy 10^{-4} as follows

$$x_2 = 8.0001, \quad x_3 = 8.0000, \quad x_4 = 8.0000.$$

Thus the largest value of number x is 8 and its corresponding y value is 2. •

Example 0.4 *Successive approximations x_n to the desired root are generated by the scheme*

$$x_{n+1} = \frac{1 + 3x_n^2}{4 + x_n^3}, \qquad n \geq 0.$$

Find $f(x_n)$ and $f'(x_n)$ and then use the Newton's method to find the approximation of the root accurate to 10^{-2}, starting with $x_0 = 0.5$.

Example 0.4 *Successive approximations x_n to the desired root are generated by the scheme*

$$x_{n+1} = \frac{1+3x_n^2}{4+x_n^3}, \qquad n \geq 0.$$

Find $f(x_n)$ and $f'(x_n)$ and then use the Newton's method to find the approximation of the root accurate to 10^{-2}, starting with $x_0 = 0.5$.

Solution. *Given*

$$x = \frac{1+3x^2}{4+x^3} = g(x),$$

and

$$x - g(x) = x - \frac{1+3x^2}{4+x^3} = \frac{x^4 - 3x^2 + 4x - 1}{4+x^3}.$$

Since

$$f(x) = x - g(x) = 0,$$

therefore, we have

$$f(x_n) = x_n^4 - 3x_n^2 + 4x_n - 1 \quad \text{and} \quad f'(x_n) = 4x_n^3 - 6x_n + 4.$$

Using these functions values in the Newton's iterative formula (2.14), we have (see Figure 2.11),

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n^2 + 4x_n - 1}{4x_n^3 - 6x_n + 4}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 0.5$, we get

$$x_1 = x_0 - \frac{x_0^4 - 3x_0^2 + 4x_0 - 1}{4x_0^3 - 6x_0 + 4} = 0.5 - \frac{0.3125}{1.5} = 0.2917.$$

Similarly, the other approximations can be obtained as

$$x_2 = 0.2917 - \frac{(-0.0813)}{2.3491} = 0.3263; \quad x_3 = 0.3263 - \frac{(-0.0029)}{2.1812} = 0.3276.$$

Example 0.5

Develop an iterative procedure for evaluating the reciprocal of a positive number N by using Newton's method. Use the developed formula to find third approximation to the reciprocal of 3, taking an initial approximation $x_0 = 0.4$. Compute absolute error.

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Solution. Consider $x = 1/N$. This problem can be easily solved by noting that we seek to find a root to the nonlinear equation

$$1/x - N = 0,$$

where $N > 0$ is the number whose reciprocal is to be found. Therefore, if $f(x) = 0$, then $x = 1/N$ is the exact root. Let

$$f(x) = 1/x - N \quad \text{and} \quad f'(x) = -1/x^2.$$

Hence, assuming an initial estimate to the root, say, $x = x_0$ and by using iterative formula (9), we get

$$x_1 = x_0 - \frac{(1/x_0 - N)}{(-1/x_0^2)} = x_0 + (1/x_0 - N)x_0^2 = x_0 + x_0 - Nx_0^2 = x_0(2 - Nx_0).$$

In general, we have

$$x_{n+1} = x_n(2 - Nx_n), \qquad n = 0, 1, \ldots, \tag{10}$$

We have to find the approximation of the reciprocal of number $N = 3$. Given the initial gauss of say $x_0 = 0.4$, then by using the iterative formula (10), we get

$$x_1 = 0.3200, \qquad x_2 = 0.3328, \qquad x_3 = 0.3333.$$

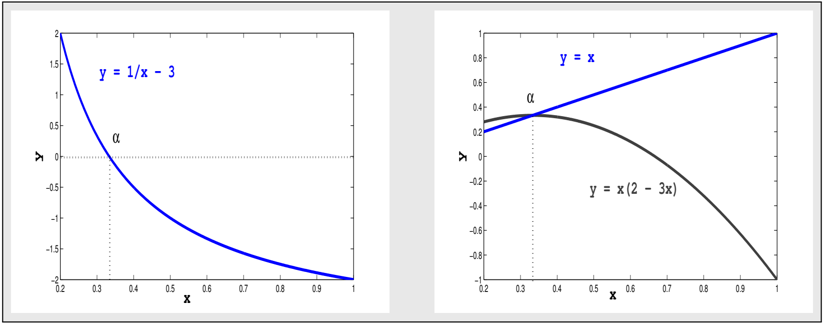

Figure 7: Graphical Solution of $1/x = 3$ and $x = x(2 - 3x)$.

After just three iterations the estimated value compares rather favorably with the exact value of $1/3 \approx 0.3333$, (see Figure 7). Thus the absolute error is

$$|E| = \left|\frac{1}{3} - x_3\right| = |0.3333 - 0.3333| = 0.0000.$$

We can calculate the other reciprocal of the number in the same way by using the general iterative formula (10).

Procedure

(Newton's Method)

1. Find the initial approximation x_0 for the root by sketching the graph of the function.
2. Evaluate function $f(x)$ and the derivative $f'(x)$ at initial approximation. Check: if $f(x_0) = 0$ then x_0 is the desire approximation to a root. But if $f'(x_0) = 0$, then go back to step *1* to choose new approximation.
3. Establish Tolerance ($\epsilon > 0$) value for the function.
4. Compute new approximation for the root by using the iterative formula (9).
5. Check Tolerance. If $|f(x_n)| \leq \epsilon$, for $n \geq 0$, then end; otherwise, go back to step *4*, and repeat the process.

Summary

In this lecture, we ...

- Introduced the Newton's Method