Numerical Methods

King Saud University

Chapter 2

Lecture #2

In this lecture, we will . . .

- Introduce the Fixed-Point Method

Fixed-Point Method

The basic idea of this method which is also called successive approximation method or function iteration, is to rearrange the original equation

$$f(x) = 0, \tag{1}$$

into an equivalent expression of the form

$$x = g(x). \tag{2}$$

Any solution of (2) is called a fixed-point for the iteration function $g(x)$ and hence a root of (1).

Definition 1

(Fixed-Point of a Function)

A *fixed-point* of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$.

For example, $x = 2$ is a fixed-point of the function $g(x) = \dfrac{x^2 - 4x + 8}{2}$ because $g(2) = 2$. •

The fixed-point method essentially solves two functions simultaneously; $y = x$ and $y = g(x)$. The point of intersection of these two functions is the solution to $x = g(x)$, and thus to $f(x) = 0$, see Figure 1.

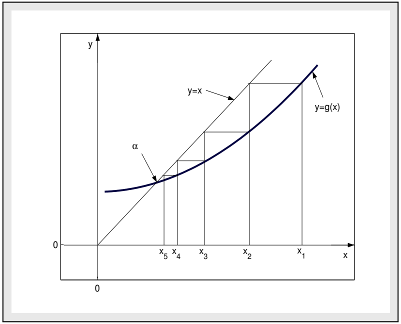


Figure 1: Graphical Solution of Fixed-Point Method.

Definition 2

(Fixed-Point Method)
The iteration defined in the following

$$x_{n+1} = g(x_n); \qquad n = 0, 1, 2, \ldots, \tag{3}$$

is called the *fixed-point method* or the *fixed-point iteration.* •

The value of the initial approximation x_0 is chosen arbitrarily and the hope is that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a number α which will automatically satisfies (1). Moreover, since (1) is a rearrangement of (2), α is guaranteed to be a zero of $f(x)$. In general, there are many different ways of rearranging of (2) in (1) form. However, only some of these are likely to give rise to successful iterations but sometime we don't have successful iterations. To describe such behaviour, we discuss the following theorem.

Basic Concept of Fixed-Point Method

Original Equation

$f(x) = 0$

Rearranged Form

$x = g(x)$

Fixed-Point

α such that $\alpha = g(\alpha)$

Iteration Formula

$x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$

The Fixed-Point Method involves rearranging an equation $f(x) = 0$ into the form $x = g(x)$. A solution to this equation is called a fixed-point of $g(x)$. The method uses an iterative formula to generate a sequence of approximations that hopefully converge to the solution.

Theorem 3

(Fixed-Point Theorem)

If g is continuously differentiable on the interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then

(a) *g has at-least one fixed-point in the given interval $[a, b]$.*

Moreover, if the derivative $g'(x)$ of the function $g(x)$ exists on an interval $[a, b]$ which contains the starting value x_0, with

$$k \equiv \max_{a \leq x \leq b} |g'(x)| < 1; \qquad \text{for all} \qquad x \in [a, b]. \tag{4}$$

Then

(b) *The sequence (3) will converge to the attractive (unique) fixed-point α in $[a, b]$.*

(c) *The iteration (3) will converge to α for any initial approximation.*

(d) *We have the error estimate*

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0|, \qquad \text{for all} \quad n \geq 1. \tag{5}$$

(e) *The limit holds:*

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha). \tag{6}$$

Fixed-Point Theorem

1 Existence

If g is continuously differentiable on $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed-point in $[a, b]$.

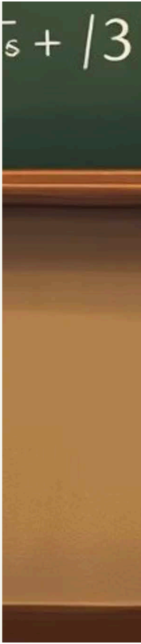
2 Convergence and uniqueness

If $|g'(x)| < 1$ for all $x \in [a, b]$, then the sequence will converge to a unique fixed-point $\alpha \in [a, b]$.

3 Error Estimate

$|\alpha - x_n| \leq \frac{k^n}{1-k} |x_1 - x_0|$, where $k = \max|g'(x)| < 1, x \in [a, b]$

The Fixed-Point Theorem provides conditions for the existence and uniqueness of a fixed-point, as well as the convergence of the iterative method. It also gives an error estimate for the approximations.



Example 0.1

Consider the nonlinear equation $x^3 = 2x + 1$ which has a root in the interval $[1.5, 2.0]$ using fixed-point method with $x_0 = 1.5$, take three different rearrangements for the equation and discuss which one is convergent or not.

Example 0.1

Consider the nonlinear equation $x^3 = 2x + 1$ which has a root in the interval $[1.5, 2.0]$ using fixed-point method with $x_0 = 1.5$, take three different rearrangements for the equation discuss which one is convergent or not.

Solution. Let us consider the three possible rearrangement of the given equation as follows:

$$
\begin{aligned}
(i) \qquad x_{n+1} &= g_1(x_n) = \frac{(x_n^3 - 1)}{2}; \qquad n = 0, 1, 2, \ldots, \\
(ii) \qquad x_{n+1} &= g_2(x_n) = \frac{1}{(x_n^2 - 2)}; \qquad n = 0, 1, 2, \ldots, \\
(iii) \qquad x_{n+1} &= g_3(x_n) = \sqrt{\frac{(2x_n + 1)}{x_n}}; \qquad n = 0, 1, 2, \ldots,
\end{aligned}
$$

then the numerical results for the corresponding iterations, starting with the initial approximation $x_0 = 1.5$ with accuracy 5×10^{-2}, are given in Table 1.

Table: Solution of $x^3 = 2x + 1$ by fixed-point method

n	x_n	$x_{n+1} = g_1(x_n) = (x_n^3 - 1)/2$	$x_{n+1} = g_2(x_n) = 1/(x_n^2 - 2)$	$x_{n+1} = g_3(x_n) = \sqrt{(2x_n + 1)/x_n}$
00	x_0	1.500000	1.500000	1.500000
01	x_1	1.187500	4.000000	1.632993
02	x_2	0.337280	0.071429	1.616284
03	x_3	-0.480816	-0.501279	1.618001
04	x_4	-0.555579	-0.571847	1.618037
05	x_5	-0.585745	-0.597731	1.618034

We note that the first two considered sequences diverge and the last one converges. This example asks the need for a mathematical analysis of the method. The following theorem gives sufficient conditions for the convergence of the fixed-point iteration. •

Now we come back to our previous Example 0.1 and discuss that why the first two rearrangements we considered, do not converge but on the other hand, last sequence has a fixed-point and converges.
Since, we observe that $f(1.5)f(2) < 0$, then the solution we seek is in the interval $[1.5, 2]$.

(i) For $g_1(x) = \dfrac{x^3 - 1}{2}$, we have $g_1'(x) = (3/2)\, x^2$, which is greater than unity throughout the interval $[1.5, 2]$. So by Fixed-Point Theorem 3 this iteration will fail to converge.

(ii) For $g_2(x) = \dfrac{1}{x^2 - 2}$, we have $g_2'(x) = \dfrac{-2x}{(x^2 - 2)^2}$, and $|g_2'(1.5)| > 1$, so from Fixed-Point Theorem 3 this iteration will fail to converge.

(iii) For $g_3(x) = \sqrt{\dfrac{2x + 1}{x}}$, we have $g_3'(x) = -x^{-3/2}/2\sqrt{2x + 1} < 1$, for all x in the given interval $[1.5, 2]$. Also, g_3 is decreasing function of x, and $g_3(1.5) = 1.63299$ and $g_3(2) = 1.58114$ both lie in the interval $[1.5, 2]$. Thus $g_3(x) \in [1.5, 2]$, for all $x \in [1.5, 2]$, so from Fixed-Point Theorem 3 the iteration will converge, see Figure 2. •

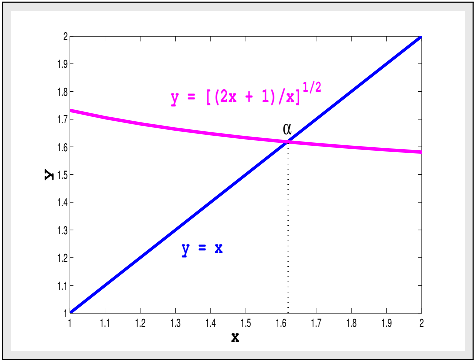


Figure 2: Graphical Solution of $x = \sqrt{(2x+1)/x}$.

Note 1

From (5) Note that the rate of convergence of the fixed-point method depends on the factor $\dfrac{k^n}{(1-k)}$; the smaller the value of k, then faster the convergence. The convergence may be very slow if the value of k is very close to 1. •

Note 2

Assume that $g(x)$ and $g'(x)$ are continuous functions of x for some open interval I, with the fixed-point α contained in this interval. Moreover assume that

$$|g'(\alpha)| < 1, \qquad \text{for} \quad \alpha \in I,$$

then, there exists an interval $[a, b]$, around the solution α for which all the conditions of Theorem 3 are satisfied. But if

$$|g'(\alpha)| > 1, \qquad \text{for} \quad \alpha \in I,$$

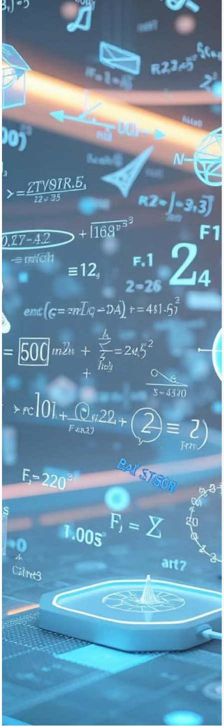
then the sequence (3) will not converge to α. In this case α is called a *repulsive fixed-point*. If

$$|g'(\alpha)| = 0, \qquad \text{for} \quad \alpha \in I,$$

then the sequence (3) converges very fast to the root α while if

$$|g'(\alpha)| = 1, \qquad \text{for} \quad \alpha \in I,$$

then the convergence the sequence (3) is not guaranteed and if the convergence happened, it would be very slow. *Thus to get the faster convergence, the value of $|g'(\alpha)|$ should be equal to zero or very close to zero.* •



Convergence Analysis

1. **Attractive Fixed-Point**

   If $|g'(\alpha)| < 1$, the sequence converges to α.

2. **Repulsive Fixed-Point**

   If $|g'(\alpha)| > 1$, the sequence will not converge to α.

3. **Fast Convergence**

   If $|g'(\alpha)| = 0$, the sequence converges very fast to α.

4. **Slow/Uncertain Convergence**

   If $|g'(\alpha)| = 1$, convergence is not guaranteed and would be very slow if it occurs.

The convergence behavior of the Fixed-Point Method depends on the derivative of $g(x)$ at the fixed-point. This analysis helps in understanding when the method will be successful and how quickly it will converge.

Example 0.2

Find an interval $[a, b]$ on which fixed-point problem $x = \dfrac{2 - e^x + x^2}{3}$ will converges. Estimate the number of iterations n within accuracy 10^{-5}.

Example 0.2

Find an interval $[a, b]$ on which fixed-point problem $x = \dfrac{2 - e^x + x^2}{3}$ will converges. Estimate the number of iterations n within accuracy 10^{-5}.

Solution. Since $x = \dfrac{2 - e^x + x^2}{3}$ can be written as

$$f(x) = e^x - x^2 + 3x - 2 = 0,$$

and we observe that $f(0)f(1) = (-1)(e^1) < 0$, then the solution we seek is in the interval $[0, 1]$.

For $g(x) = \dfrac{2 - e^x + x^2}{3}$, we have $g'(x) = \dfrac{2x - e^x}{3} < 1$, for all x in the given interval $[0, 1]$. Also, g is decreasing function of x and $g(0) = 0.3333$ and $g(1) = \dfrac{3 - e}{3} = 0.0939$ both lie in the interval $[0, 1]$. Thus $g(x) \in [0, 1]$, for all $x \in [0, 1]$, so from Fixed-Point Theorem 3 the g(x) has a unique fixed-point in $[0, 1]$. Taking $x_0 = 0.5$, we have

$$x_1 = g(x_0) = \frac{2 - e^{x_0} + x_0^2}{3} = 0.2004.$$

Also, we have

$$k_1 = |g'(0)| = 0.3333 \quad \text{and} \quad k_2 = |g'(1)| = 0.2394,$$

which give $k = \max\{k_1, k_2\} = 0.3333$. Thus the error estimate (5) within the accuracy 10^{-5} is

$$|\alpha - x_n| \leq 10^{-5}, \qquad \text{gives} \qquad \frac{(0.3333)^n}{1 - 0.3333}(0.2996) \leq 10^{-5},$$

and by solving this inequality, we obtain $n \leq 9.7507$. So we need ten approximations to get the desired accuracy for the given problem. •

Example 0.3

Show that the function $g(x) = 3^{-x}$ on the interval $[0, 1]$ has at least one fixed-point but it is not unique.

Example 0.3

Show that the function $g(x) = 3^{-x}$ on the interval $[0, 1]$ has at least one fixed-point but it is not unique.

Solution. Given $x = g(x) = 3^{-x}$, and it can be written as

$$x - 3^{-x} = f(x) = 0.$$

So $f(0)(1) = (-1)(2/3) < 0$, so $f(x)$ has a root in the interval $[0, 1]$, see Figure 3. Note that g is decreasing function of x and $g(0) = 1$ and $g(1) = 0.3333$ both lie in the interval $[0, 1]$. Thus $g(x) \in [0, 1]$, for all $x \in [0, 1]$, so from Fixed-Point Theorem 3 the function $g(x)$ has at least one fixed-point in $[0, 1]$. Since the derivative of the function $g(x)$ is

$$g'(x) = -3^{-x} \ln 3,$$

which is less than zero on $[0, 1]$, therefore, the function g is decreasing on $[0, 1]$. But $g'(0) = -\ln 3 = -1.0986$, so

$$|g'(x)| > 1 \quad \text{on} \qquad (0, 1).$$

Thus from Fixed-Point Theorem 3 the function $g(x)$ has no unique fixed-point in $[0, 1]$. •

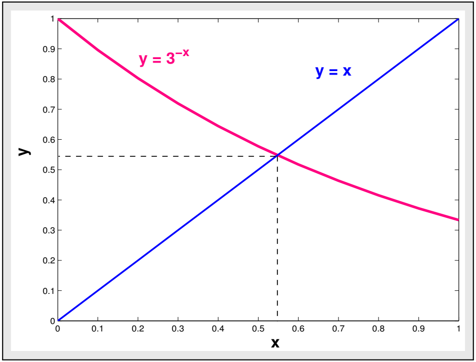


Figure 3: Graphical Solution of $x = 3^{-x}$.

Example 0.4

Show that the function $g(x) = \sqrt{2x - 1}$ on the interval $[0, 1]$ that satisfies none of the hypothesis of Theorem 3 but still has a unique fixed-point on $[0, 1]$.

Example 0.4

Show that the function $g(x) = \sqrt{2x-1}$ on the interval $[0,1]$ that satisfies none of the hypothesis of Theorem 3 but still has a unique fixed-point on $[0,1]$.

Solution. Since $x = g(x) = \sqrt{2x-1}$, it gives

$$x^2 - 2x + 1 = (x-1)^2 = f(x) = 0.$$

Then $x = \alpha = 1 \in [0,1]$ is the root of the nonlinear equation $f(x) = 0$ and the fixed-point of the function $g(x)$ as $g(1) = 1$. But notice that the function $g(x)$ is not continuous on the interval $[0,1]$ and the derivative of the function $g(x)$

$$g'(x) = \frac{1}{\sqrt{2x-1}},$$

does not exist on the interval $(0,1)$. So all the conditions of Fixed-Point Theorem 3 fail. •

Example 0.5

Show that the fixed point form of the equation $x = N^{1/3}$ can be written as $x = Nx^{-2}$ and the associated iterative scheme

$$x_{n+1} = Nx_n^{-2}, \qquad n \geq 0,$$

will not successful (diverge) in finding the approximation of cubic root of the positive number N.

Example 0.5

Show that the fixed point form of the equation $x = N^{1/3}$ can be written as $x = Nx^{-2}$ and the associated iterative scheme

$$x_{n+1} = Nx_n^{-2}, \qquad n \geq 0,$$

will not successful (diverge) in finding the approximation of cubic root of the positive number N.

Solution. Given $x = N^{1/3}$ and it can be written as

$$x^3 - N = 0 \quad \text{or} \quad x = \frac{N}{x^2} = Nx^{-2}.$$

It gives the iterative scheme

$$x_{n+1} = Nx_n^{-2} = g(x_n), \qquad n \geq 0.$$

From this, we have

$$g(x) = Nx^{-2} \qquad \text{and} \qquad g'(x) = -2Nx^{-3}.$$

Since $\alpha = x = N^{1/3}$, therefore

$$g'(\alpha) = -2N\alpha^{-3} \quad \text{and} \quad g'(N^{1/3}) = -2N(N^{1/3})^{-3} = -2NN^{-1} = -2.$$

Thus

$$|g'(N^{1/3})| = |-2| = 2 > 1,$$

which shows the divergence. •

Example 0.6

One of the possible rearrangement of the nonlinear equation $e^x = x + 2$, which has root in $[1, 2]$ is

$$x_{n+1} = g(x_n) = \ln(x_n + 2); \qquad n = 0, 1, \ldots.$$

(a) Show that $g(x)$ has a unique fixed-point in $[1, 2]$.

(b) Use fixed-point iteration formula (3) to compute approximation x_3, using $x_0 = 1.5$.

(c) Compute an error estimate $|\alpha - x_3|$ for your approximation.

(d) Determine the number of iterations needed to achieve an approximation with accuracy 10^{-2} to the solution of $g(x) = \ln(x + 2)$ lying in the interval $[1, 2]$ by using the fixed-point iteration method.

Solution. Since, we observe that $f(1)f(2) < 0$, then the solution we seek is in the interval $[1, 2]$.

(a) For $g(x) = \ln(x+2)$, we have $g'(x) = 1/(x+2) < 1$, for all x in the given interval $[1, 2]$. Also, g is increasing function of x, and $g(1) = \ln(3) = 1.0986123$ and $g(2) = \ln(4) = 1.3862944$ both lie in the interval $[1, 2]$. Thus $g(x) \in [1, 2]$, for all $x \in [1, 2]$, so from fixed-point theorem the g(x) has a unique fixed-point, see Figure 4.

(b) using the given initial approximation $x_0 = 1.5$, we have the other approximations as

$$x_1 = g(x_0) = 1.252763, \quad x_2 = g(x_1) = 1.179505, \quad x_3 = g(x_2) = 1.156725.$$

(c) Since $a = 1$ and $b = 2$, then the value of k can be found as follows

$$k_1 = |g'(1)| = |1/3| = 0.333 \quad \text{and} \quad k_2 = |g'(2)| = |1/4| = 0.25,$$

which give $k = \max\{k_1, k_2\} = 0.333$. Thus using the error formula (5), we have

$$|\alpha - x_3| \leq \frac{(0.333)^3}{1 - 0.333} |1.252763 - 1.5| = 0.013687.$$

(d) From the error bound formula (5), we have

$$\frac{k^n}{1-k} |x_1 - x_0| \leq 10^{-2}.$$

By using above parts *(b)* and *(c)*, we have

$$\frac{(0.333)^n}{1 - 0.333} |1.252763 - 1.5| \leq 10^{-2}.$$

Solving this inequality, we obtain

$$n \ln(0.333) \leq \ln(0.02698), \quad \text{gives}, \quad n \geq 3.28539.$$

So we need four approximations to get the desired accuracy for the given problem.

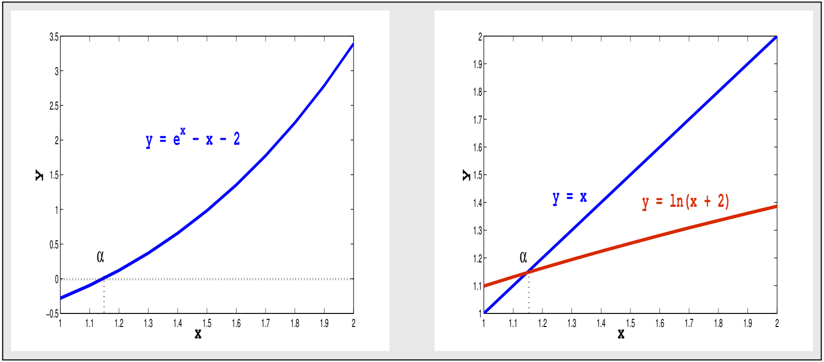


Figure 4: Graphical Solution of $e^x = x + 2$ Graphical solution of $x = \ln(x + 2)$.

Example 0.7 *Which of the following sequences will converge faster to* $\sqrt{5}$

$$(a) \quad x_{n+1} = x_n + 1 - \frac{x_n^2}{5}, \qquad (b) \quad x_{n+1} = \frac{1}{3}\left[3x_n + 1 - \frac{x_n^2}{5}\right].$$

Example 0.7 *Which of the following sequences will converge faster to* $\sqrt{5}$

$$(a) \quad x_{n+1} = x_n + 1 - \frac{x_n^2}{5}, \qquad (b) \quad x_{n+1} = \frac{1}{3}\left[3x_n + 1 - \frac{x_n^2}{5}\right].$$

Solution. *It can be easily verify by using the Note 2.2. From the first sequence, we have*

$$g_1(x) = x + 1 - \frac{x^2}{5} \quad \text{and} \quad g_1'(x) = 1 - \frac{2x}{5},$$

which implies that

$$|g_1'(\sqrt{5})| = \left|1 - \frac{2\sqrt{5}}{5}\right| = 0.1056 < 1.$$

Similarly, from the second sequence, we have

$$g_2(x) = \frac{1}{3}\left[3x + 1 - \frac{x^2}{5}\right] \quad \text{and} \quad g_2'(x) = \frac{1}{3}\left[3 - \frac{2x}{5}\right], \quad \text{gives,} \quad |g_2'(\sqrt{5})| = 0.701186 < 1.$$

We note that both sequences are converging to $\sqrt{5}$ *but the sequence (a) will converges faster than the sequence (b) because the value of* $|g_1'(\sqrt{5})|$ *is closer to zero than by* $|g_2'(\sqrt{5})|$. •

Procedure
(Fixed-Point Method))

1. Choose an initial approximation x_0 such that $x_0 \in [a, b]$.
2. Choose a convergence parameter $\epsilon > 0$.
3. Compute new approximation x_{new} by using the iterative formula (3).
4. Check, if $|x_{new} - x_0| < \epsilon$ then x_{new} is the desire approximate root; otherwise set $x_0 = x_{new}$ and go to step *3*.

Summary

In this lecture, we ...

- Introduced the Fixed-Point Method