Numerical Methods

King Saud University

Chapter 2

Lecture #1

In this lecture, we will . . .

- Considering how to numerically find roots of algebraic equations
- Introduce the Bisection method

Types of Nonlinear Equations

1) Polynomial Equations

Equations of degree $n > 1$, such as

$$x^2 + 5x + 6 = 0$$

$$x^3 = 2x + 1$$

$$x^{200} - 2x + 1 = 0$$

2) Non-Polynomial Equations

The power of the unknown variable (not a positive integer number)

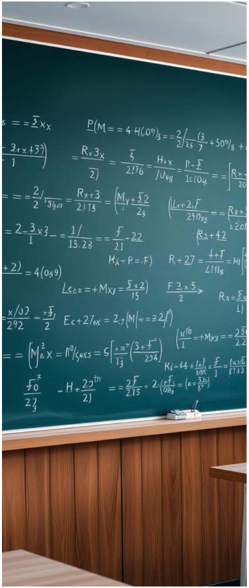
$$x^{-1} + 2x = 1 \quad ; \quad \sqrt{x} + x = 2$$

$$x^{\frac{2}{3}} + \frac{2}{x} + 4 = 0$$

3) Equations involving trigonometric, exponential, or logarithmic functions

$$x = \cos(x) \quad ; \quad e^x + x - 10 = 0 \quad ; \quad x + \ln x = 10$$

Important Points



Roots of Nonlinear Equations

1

Definition

A root α is a solution to the equation $f(x) = 0$, also called a zero of the function f .

2

Continuity

If the function $f(x)$ is continuous on the interval $[a, b]$ and the values $f(a)$ and $f(b)$ have opposite signs, then a root must exist somewhere within that interval. This is known as the Intermediate Value Theorem, and it provides a powerful tool for locating roots.

3

Types of Roots

▼ Roots can be simple or multiple.

> A simple root has $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, crossing the x -axis.
>
> A multiple root has $f(\alpha) = 0$ and $f'(\alpha) = 0$ tangent to the x -axis.

Important Points

- Simple root means

$$f(\alpha) = 0 \text{ but } f'(\alpha) \neq 0.$$

For example, $\alpha_1 = -3$ and $\alpha_2 = -2$ are the simple roots of the nonlinear equation

$x^2 + 5x + 6 = 0.$

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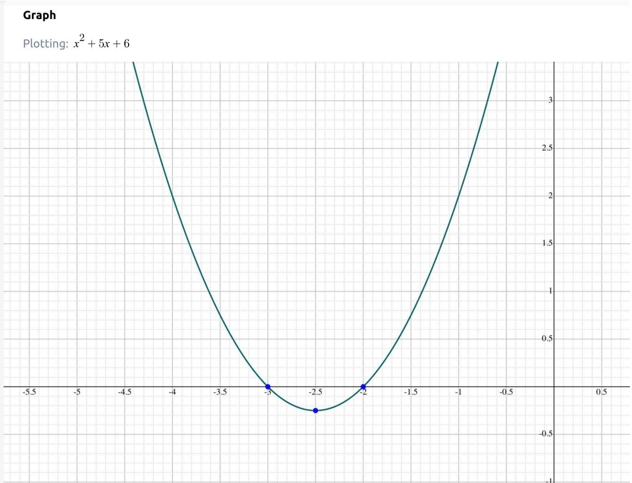

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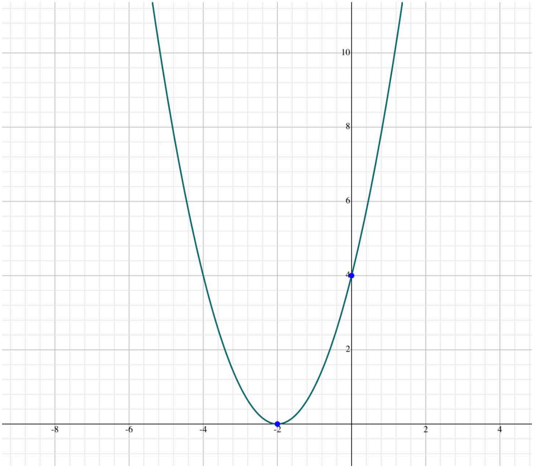
Graph

Plotting: $x^2 + 4x + 4$



Important Points

Iterative Methods for Nonlinear Equations

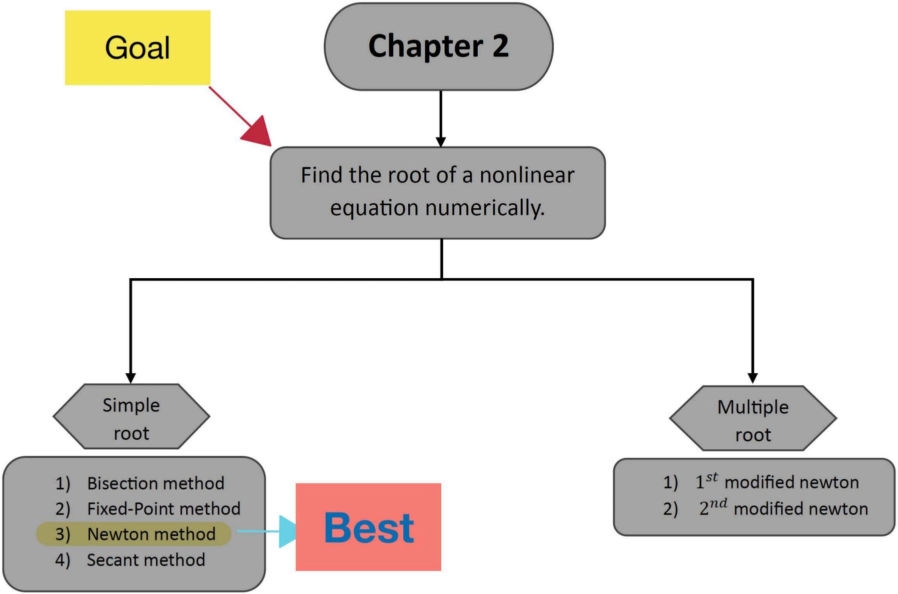
There are several iterative methods we can use to approximate the roots (solutions) of nonlinear equations. These include the bisection method, fixed-point method, Newton's method, and the secant method. These methods can find approximations for single or simple roots of nonlinear equations.

For equations with multiple roots, we can use other iterative methods like the first modified Newton's method (also called Schroeder's method) and the second modified Newton's method.

Best Methods

The best method for approximating a simple root of a nonlinear equation is Newton's method, *which is a quadratic convergent method*. For multiple roots, the modified Newton's method is also *a quadratic convergent method*,

Newton's method for multiple roots is a linear convergent method.


Goal
Chapter 2
Find the root of a nonlinear equation numerically.
Simple root
1) Bisection method
2) Fixed-Point method
3) Newton method
4) Secant method
Best
Multiple root
1) 1^{st} modified newton
2) 2^{nd} modified newton

The Bisection Method

The Bisection method is used to determine, to any specified accuracy that your computer will permit, a solution to $f(x) = 0$ on an interval $[a, b]$, provided:

- $f(x)$ is continuous on $[a, b]$;
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The concept of the Bisection method is simple, and is based on utilizing the Intermediate Value Theorem. Essentially, due the continuity of f on $[a, b]$, and since $f(a)f(b) < 0$, then there must be a point $a < \alpha < b$ such that $f(\alpha) = 0$. The implication is that one of the values is negative and the other is positive. These conditions can be easily satisfied by sketching the function, see Figure 1.

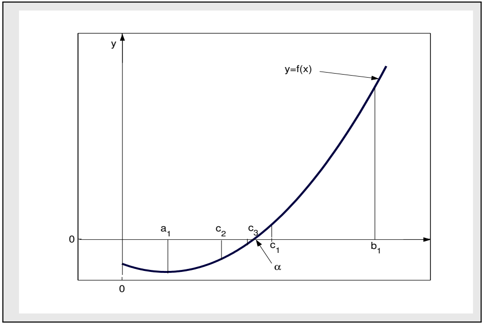

Figure: Graphical Solution of Bisection Method.

Therefore the root must lies between a and b (by Intermediate Value Theorem) and a new approximation to the root α be calculated as

$$c = \frac{a+b}{2},$$

and, in general

$$c_n = \frac{a_n + b_n}{2}, \qquad n \geq 1. \tag{2}$$

The iterative formula (2) is known as the **bisection method**.

If $f(c) \approx 0$, then $c \approx \alpha$ is the desired root, and, if not, then there are two possibilities.

- Firstly, if $f(a)f(c) < 0$, then $f(x)$ has a zero between point a and point c. The process can then be repeated on the new interval $[a, c]$.
- Secondly, if $f(a)f(c) > 0$ it follows that $f(b)f(c) < 0$ since it is known that $f(b)$ and $f(c)$ have opposite signs. Hence, $f(x)$ has zero between point c and point b and the process can be repeated with $[c, b]$. We see that after one step of the process, we have found either a zero or a new bracketing interval which is precisely half the length of the original one.
- The process continue until the desired accuracy is achieved.

1) **Write** $f(x) = 0$

2) **Check that** $f(a) \, . \, f(b) < 0$

3) **The 1st approximation** $c_1 = \frac{a+b}{2}$ **, and find** $f(c_1)$

4) **To find the 2nd approximation, we do the following**

If $f(a) \, . \, f(c_1) < 0$

Then $c_2 \in [a, c_1]$

$$c_2 = \frac{a + c_1}{2} \rightarrow\rightarrow f(c_2)$$

If $f(c_1) \, . \, f(b) < 0$

Then $c_2 \in [c_1, b]$

$$c_2 = \frac{c_1 + b}{2} \rightarrow\rightarrow f(c_2)$$

⋮

etc

Example 0.1

Use the bisection method to find the approximation to the root of the equation

$$x^3 = 2x + 1,$$

that is located in the interval [1.5, 2.0] accurate to within 10^{-2}.

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Solution. Since the given function $f(x) = x^3 - 2x - 1$ is a polynomial function and so is continuous on $[1.5, 2.0]$, starting with $a_1 = 1.5$ and $b_1 = 2$, we compute:

$$\begin{array}{lcll} a_1 & = & 1.5: & f(a_1) = -0.625 \\ b_1 & = & 2.0: & f(b_1) = 3.0, \end{array}$$

and since $f(1.5)f(2.0) < 0$, so that a root of $f(x) = 0$ lies in the interval $[1.5, 2.0]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 1.75; \qquad f(c_1) = 0.859375.$$

Hence the function changes sign on $[a_1, c_1] = [1.5, 1.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the midpoint is:

$$c_2 = \frac{a_2 + b_2}{2} = 1.625; \qquad f(c_2) = 0.041056.$$

Continue in this way we obtain a sequence $\{c_k\}$ of approximation shown by Table 1.

Table: Solution of $x^3 = 2x + 1$ by bisection method

n	Left Endpoint a_n	Midpoint c_n	Right Endpoint b_n	Function Value $f(c_n)$
01	1.500000	1.750000	2.000000	0.8593750
02	1.500000	1.625000	1.750000	0.0410156
03	1.500000	1.562500	1.625000	-0.3103027
04	1.562500	1.593750	1.625000	-0.1393127
05	1.593750	1.609375	1.625000	-0.0503273
06	1.609375	1.617188	1.625000	-0.0049520

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We see that the functional values are approaching zero as the number of iterations is increase. We got the desired approximation to the root of the given equation is $c_6 = 1.617188 \approx \alpha$ after 6 iterations with accuracy $\epsilon = 10^{-2}$. •

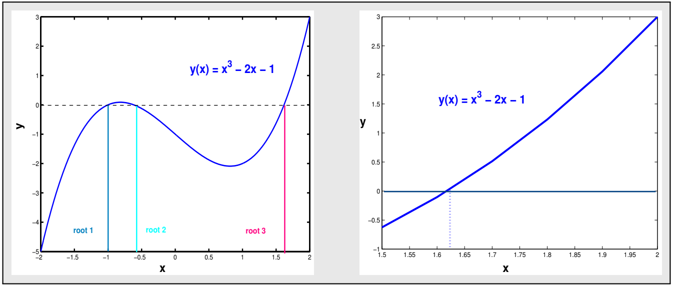

Figure: Graphical Solution of $x^3 = 2x + 1$ in the intervals $[-2, 2]$ and $[1.5, 2]$.

Example 0.2

Find the point of intersection of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$, then use bisection method within accuracy 10^{-3}.

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Solution. The graphs in the Figure 3 show that there is an intersection at about point $(0.66, 0.61)$.

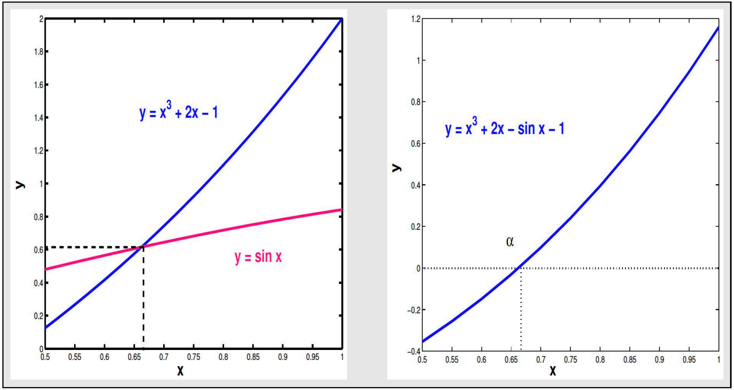

Figure: Graphical Solution of $\sin x = x^3 + 2x - 1$ and $x^3 + 2x - \sin x = 1$.

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Solution. The graphs in the Figure 3 show that there is an intersection at about point $(0.66, 0.61)$. Using the function $f(x) = x^3 + 2x - \sin x - 1$ and the starting interval $[0.5, 1.0]$, we compute:

$$\begin{array}{rcll} a_1 & = & 0.5: & f(a_1) = -0.3544, \\ b_1 & = & 1.0: & f(b_1) = 1.1585. \end{array}$$

Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5).f(1.0) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1.0]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \qquad f(c_1) = 0.240236.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the midpoint is:

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \qquad f(c_2) = -0.090957.$$

Then continue in this manner we obtain a sequence of approximation shown by the next Table

Table: Solution of $x^3 + 2x - \sin x - 1$ by bisection method

n	Left Endpoint a_n	Right Endpoint b_n	Midpoint c_n	Function Value $f(c_n)$
01	0.5000	1.0000	0.750000	0.240236
02	0.5000	0.7500	0.625000	-0.090957
03	0.6250	0.7500	0.687500	0.065344
⋮	⋮	⋮	⋮	⋮
07	0.6563	0.6641	0.660156	-0.005228
08	0.6602	0.6641	0.662109	-0.000302

We see that the functional values are approaching zero as the number of iterations is increase. We got the desired approximation to the root of the given equation is $c_8 = 0.662109 \approx \alpha$ after 8 iterations with accuracy $\epsilon = 10^{-3}$. •

Theorem 1

(Bisection Convergence and Error Theorem)

Let $f(x)$ be continuous function defined on the given initial interval $[a_0, b_0] = [a, b]$ and suppose that $f(a)f(b) < 0$. Then bisection method (2) generates a sequence $\{c_n\}_{n=1}^{\infty}$ approximating $\alpha \in (a, b)$ with the property

$$|\alpha - c_n| \leq \frac{b-a}{2^n}, \quad n \geq 1. \tag{3}$$

Moreover, to obtain accuracy of

$$|\alpha - c_n| \leq \epsilon,$$

(for $\epsilon = 10^{-k}$) it suffices to take

$$n \geq \frac{\ln\left\{10^k(b-a)\right\}}{\ln 2}, \tag{4}$$

where k is nonnegative integer.

Note:

The above Theorem 1 gives us information about bounds for errors in approximation and the number of bisections needed to obtain any given accuracy.

Example 0.3

Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $xe^x = 1$ lying in the interval $[0.5, 1]$ using the bisection method. Find an approximation to the root with this degree of accuracy.

Example 0.3

Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $xe^x = 1$ lying in the interval $[0.5, 1]$ using the bisection method. Find an approximation to the root with this degree of accuracy.
Solution. Here $a = 0.5$, $b = 1$ and $k = 1$, then by using inequality (4), we get

$$n \geq \frac{\ln[10^1(1 - 0.5)]}{\ln 2} \approx 2.3219.$$

So no more than three iterations are required to obtain an approximation accurate to within 10^{-1}.
The given function $f(x) = xe^x - 1$ is continuous on $[0.5, 1.0]$, so starting with $a_1 = 0.5$ and $b_1 = 1$, we compute:

$$\begin{array}{lcll} a_1 & = & 0.5: & f(a_1) = -0.1756, \\ b_1 & = & 1: & f(b_1) = 1.7183, \end{array}$$

since $f(0.5)f(1) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \qquad f(c_1) = 0.5878.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the bisection formula gives

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \qquad f(c_2) = 0.1677.$$

Finally, we have in the similar manner as

$$c_3 = \frac{a_3 + b_3}{2} = 0.5625, \qquad f(c_3) = 0.01.$$

the value of the third approximation which is accurate to within 10^{-1}. •

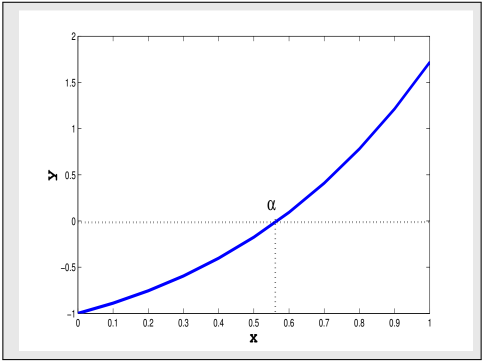

Figure: Graphical Solution of $xe^x = 1$.

Example 0.4

Use the bisection method to compute the first three approximate values for $\sqrt[4]{18}$. Also, compute an error bound and absolute error for your approximation.

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Use the bisection method to compute the first three approximate values for $\sqrt[4]{18}$. Also, compute an error bound and absolute error for your approximation.
Solution. Consider

$$x = \sqrt[4]{18} = (18)^{1/4}, \qquad \text{or} \qquad x^4 - 18 = 0.$$

Choose the interval $[2, 2.5]$ on which the function $f(x) = x^4 - 18$ is continuous and the function $f(x)$ satisfies the sign property, that is

$$f(2)f(2.5) = (-2)(21.0625) = -42.125 < 0.$$

Hence root $\alpha = \sqrt[4]{18} = 2.0598 \in [2, 2.5]$ and we compute its first approximate value by using formula (2) (when $n = 1$) as follows:

$$c_1 = \frac{2.0 + 2.5}{2} = 2.2500 \quad \text{and} \quad f(2.25) = 7.6289.$$

Since the function $f(x)$ changes sign on $[2.0, 2.25]$. To continue, we squeeze from right and use formula (2) again to get the following second approximate value of the root α as:

$$c_2 = \frac{2.0 + 2.25}{2} = 2.1250 \quad \text{and} \quad f(2.1250) = 2.3909.$$

Then continue in the similar way, the third approximate value of the root α is $c_3 = 2.0625$ with $f(2.0625) = 0.0957$.

Note that the value of the function at each new approximate value is decreasing which shows that the approximate values are coming closer to the root α. Now to compute the error bound for the approximation we use the formula (3) and get

$$|\alpha - c_3| \leq \frac{2.5 - 2.0}{2^3} = 0.0625,$$

which is the possible maximum error in our approximation and

$$|E| = |2.0598 - 2.0625| = 0.0027,$$

be the absolute error in the approximation. •

Solved previous exam questions

Q1: Given that $f(x) = x^2 - 4x + 4 - \ln x = 0, \quad x \in [2, 4]$

(a) Use the Bisection method to find c_1 and c_2.

(b) Find the number of iterations needed to achieve an approximation with accuracy 10^{-5}.

(a)

$$f(x) = x^2 - 4x + 4 - \ln x = 0$$

$$c = \frac{a+b}{2},$$

$a_1 = 2 \longrightarrow f(a_1) =$ -0.6931

$b_1 = 4 \longrightarrow f(b_1) =$ 2.6137

$c_1 = \frac{a_1 + b_1}{2} = 3 \longrightarrow f(c_1) =$ -0.0986

$c_2 = \frac{a_2 + b_2}{2} = 3.5 \longrightarrow f(c_2) =$ 0.9972

(b)

$$|\alpha - c_n| \leq \frac{b-a}{2^n}, \quad n \geq 1. \longrightarrow n \geq \frac{\ln\left\{10^k (b-a)\right\}}{\ln 2},$$

where k is nonnegative integer.

$n \approx 17.6 \longrightarrow n = 18$

Estimate the number of iterations required to achieve an approximation with accuracy 10^{-5} to the solution of $x^2 = 3$ lying in the interval $[1, 2]$ using the bisection method. Also, **compute** the first three approximations c_1, c_2 and c_3 [use 4 decimal places in calculations].

$f(x) = x^2 - 3 \Rightarrow f(1) = -2 < 0$ and $f(2) = 1 > 0$. Thus, according to the Intermediate Value Theorem, there must be a point $c_n \in [1,2]$ such that $f(c_n) = 0$.

Now, let $a = 1,\ b = 2$, so

$$c_1 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5 \quad ---> \quad f(1.5) = -0.75 < 0$$

$$c_2 = \frac{1.5+2}{2} = 1.75 \quad ---> \quad f(1.75) = 0.0625 > 0$$

$$c_3 = \frac{1.5+1.75}{2} = 1.625$$

Now, we find the number of iterations required to achieve an approximation with accuracy

10^{-5} as follows:

$$\frac{b-a}{2^n} \leq 10^{-5} \ --> \ \frac{2-1}{2^n} \leq 10^{-5} \ --> \ 2^{-n} \leq 10^{-5} \ --> \ n \geq \frac{5\ln 10}{\ln 2} \approx 16.6096$$

So, the number of iterations required to achieve an approximation with accuracy 10^{-5} is 17

Procedure
(Bisection Method)

1. Establish an interval $a \leq x \leq b$ such that $f(a)$ and $f(b)$ are of opposite sign, that is, $f(a).f(b) < 0$.
2. Choose an error tolerance ($\epsilon > 0$) value for the function.
3. Compute a new approximation for the root:
$$c_n = \frac{(a_n + b_n)}{2}; \qquad n = 1, 2, 3, \ldots.$$
4. Check tolerance. If $|f(c_n)| \leq \epsilon$, use $c_n,\ n \geq 1$ for desired root; otherwise continue.
5. Check, if $f(a_n)f(c_n) < 0$, then set $b_n = c_n$; otherwise set $a_n = c_n$.
6. Go back to step *3*, and repeat the process.

Summary

In this lecture, we ...

- Considered how to numerically find roots of algebraic equations
- Introduced the Bisection method.