Multiple Compounding Periods in a Year

Example 2.36

Rebecca Carlson purchased a car for $25,000 by borrowing the money at *8% per year compounded monthly*. She paid off the loan with 60 equal monthly payments, the first of which was paid one month after receiving the car. How much was her monthly payment?

Solution

A = $25,000(A|P ⅔%,60) = $25,000(0.02028) = $507.00

A =PMT(0.08/12,60,-25000)

A = $506.91

Compounding and Timing

The effective annual interest rate (i_{eff}) is used to deal with differences in the timing of cash flows and the compounding frequency.

$$i_{eff} = \left(1 + \frac{r}{m}\right)^{m} - 1 = \left(F \mid P \ \frac{r\%}{m}, m\right) - 1$$

- **r = nominal annual interest rate**
- **m = number of compound periods per year**

- **i_{eff} =EFFECT(r,m)**

Example 2.37

What is the effective interest rate for 12% per annum *compounded annually*?

r = 12%, m = 1, r/m = 12%

i_{eff} = (F|P 12%,1)-1 = 1.1200-1.0 = 12.00%

=EFFECT(12%,1) = 12%

What is the effective interest rate for 12% per annum *compounded semiannually*?

r = 12%, m = 2, r/m = 6%

i_{eff} = (F|P 6%,2)-1 = 1.1236-1.0 = 12.36%

=EFFECT(12%,2) = 12.36%

What is the effective interest rate for 12% per annum *compounded quarterly*?

$r = 12\%$, $m = 4$, $r/m = 3\%$

$i_{eff} = (F|P\ 3\%,4)-1 = 1.12551-1.0 = 12.551\%$

=EFFECT(12%,4) = 12.551%

What is the effective interest rate for 12% per annum *compounded monthly*?

$r = 12\%$, $m = 12$, $r/m = 1\%$

$i_{eff} = (F|P\ 1\%,12)-1 = 1.12683-1.0 = 12.683\%$

=EFFECT(12%,12) = 12.683%

What is the effective interest rate for 12% per annum *compounded weekly*?

r = 12% m = 52,

$i_{eff} = (1 + 0.12/52)^{52} - 1 = 1.1273409872 - 1.0$

$= 12.73409872\%$

=EFFECT(12%,52) = 12.73409872%

What is the effective interest rate for 12% per annum *compounded daily*?

r = 12% m = 365

$i_{eff} = (1 + 0.12/365)^{365} - 1 = 1.1274746156 - 1.0$

$= 12.74746156\%$

=EFFECT(12%,365) = 12.74746156%

What is the effective interest rate for 12% per annum *compounded hourly*?

$r = 12\%$ $\quad\quad$ $m = 8760$,

$i_{eff} = (1 + 0.12/8760)^{8760} - 1 = 1.12749592487 82 - 1.0$

$= 12.74959248782\%$

=EFFECT(12%,8760) = 12.74959248776% (!!!)

What is the effective interest rate for 12% per annum *compounded every minute*?

$r = 12\%$ $\quad\quad$ $m = 525{,}600$,

$i_{eff} = (1 + 0.12/525{,}600)^{525{,}600} - 1 = 1.127496836146 - 1.0$

$= 12.7496836146\%$

=EFFECT(12%,525600) = 12.7496836133% (!!!)

What is the effective interest rate for 12% per annum *compounded every second*?

$r = 12\%$ $m = 31{,}536{,}000$

$i_{eff} = (1 + 0.12/31{,}536{,}000)^{31{,}536{,}000} - 1 = 1.127496851431 - 1.0$

$= 12.7496851431\%$

=EFFECT(12%,31536000) = 12.7496852242% (!!!)

What is the effective interest rate for 12% per annum *compounded continuously*?

$r = 12\%$ m = infinitely large

From Appendix 2A, we'll find that

$i_{eff} = e^{0.12} - 1 = 12.7496851579\%$

Shorthand Notation

8% compounded quarterly is denoted 8%/year/quarter

and 12% per annum compounded monthly is denoted 12%/year/month.

Example 2.38

Wenfeng Li borrowed \$1,000 and paid off the loan in 4.5 years with a single lump sum payment of \$1,500. Based on a 6-month interest period, what was the *annual effective interest rate* paid?

n = 9 6-mo periods, P = \$1,000, F = \$1,500, i = ?

\$1,500 = \$1,000(F|P i%,9)

$(1 + i)^9 = 1.5$ or i = 0.046 or 4.6%

$i_{eff} = (1 + 0.046)^2 - 1 = 0.0943$ or 9.43%

i_{eff}=RATE(4.5,,-1000,1500) = 9.429%

Example 2.39

Greg Wilhelm borrowed $100,000 to purchase a house. It will be repaid with 360 equal monthly payments at a nominal annual rate of *6% (6%/year/month).* Determine the monthly payments.

A = $100,000(*A*/*P* 0.5%,360)

= $100,000(0.0059955)

= $599.55/month

=PMT(0.06/12,360,-100000)

= $599.55

Example 2.39 CONT.

In addition to the usual interest charges, Mr. Wilhelm had to pay \$2,000 in closing costs to the lending firm. If the closing costs are financed, what would be the size of the monthly payments?

A = \$102,000(*A*/*P* 0.5%,360)

= \$102,000(0.0059955)

= \$611.54/month

=PMT(0.06/12,360,-102000)

= \$611.54

When Compounding and Cash Flow Frequencies Differ

When compounding frequency and cash flow frequency differ, the following approach is taken.

Let *r* denote the nominal annual interest rate for money and

. *m* denote the number of compounding periods in a year; and

***k* denote number of *cash flows in a year* and**

***i* denote the *interest rate per cash flow period*.**

The value of *i* is obtained as follows,

$$i = (1 + r/m)^{m/k} - 1 \qquad (2.49)$$

Equation 2.49 results from setting the effective annual interest rate for the stated compounding frequency of money equal to the effective annual interest rate for the cash flow frequency.

$$(1 + i)^k - 1 = (1 + r/m)^m - 1$$

and solving for *i*.

Example 2.40

What size ***monthly*** payments should occur when $10,000 is borrowed **at 8% compounded quarterly (8%/year/quarter)** and the loan is repaid with ***36 equal monthly payments***?

From Equation 2.49, $r = 0.08$, $m = 4$, and $k = 12$. Therefore,

$i = (1 + 0.08/4)^{4/12} – 1 = 0.006623$ or ***0.6623%/month***

Knowing the monthly interest rate, the monthly payment can be determined,

$A = \$10,000(A/P\ 0.6623\%,36)$

$= \$10,000[(0.006623)(1.006623)^{36}]/[(1.006623)^{36} - 1]$

$= \$313.12$

Using the Excel® **PMT** worksheet function,

A **=PMT(1.02^(1/3)-1,36,-10000)**

= $313.12

Summary of discrete compounding interest factors.

To Find	Given	Factor	Symbol	Name
P	F	$(1+i)^{-n}$	$(P\mid F\ i\%,n)$	Single sum, present worth factor
F	P	$(1+i)^{n}$	$(F\mid P\ i\%,n)$	Single sum, compound amount factor
P	A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$(P\mid A\ i\%,n)$	Uniform series, present worth factor
A	P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$(A\mid P\ i\%,n)$	Uniform series, capital recovery factor
F	A	$\frac{(1+i)^n - 1}{i}$	$(F\mid A\ i\%,n)$	Uniform series, compound amount factor
A	F	$\frac{i}{(1+i)^n - 1}$	$(A\mid F\ i\%,n)$	Uniform series, sinking fund factor
P	G	$\frac{[1-(1+ni)(1+i)^{-n}]}{i^2}$	$(P\mid G\ i\%,n)$	Gradient series, present worth factor
A	G	$\frac{(1+i)^n - (1+ni)}{i[(1+i)^n - 1]}$	$(A\mid G\ i\%,n)$	Gradient series, uniform series factor
P	A_{1},j	$\frac{1-(1+j)^n(1+i)^{-n}}{i-j}$ for $i \neq j$	$(P\mid A_1\ i\%,j\%,n)$	Geometric series, present worth factor
F	A_{1},j	$\frac{(1+i)^n - (1+j)^n}{i-j}$ for $i \neq j$	$(F\mid A_1\ i\%,j\%,n)$	Geometric series, future worth factor

Summary of Selected Excel® Worksheet Financial Functions

To Find	Given	Worksheet Function	Cell Entry
P	i, n, F	PV	=PV(i%,n,,-F)
P	i, n, A	PV	=PV(i%,n,-A)
P	i, n, F, A	PV	=PV(i%,n,-A,-F)
P	$i, A_1, A_2, \ldots, A_n$	NPV	=NPV(i%,A_1,A_2,…,A_n)
F	i, n, P	FV	=FV(i%,n,,-P)
F	i, n, A	FV	=FV(i%,n,-A)
F	i, n, P, A	FV	=FV(i%,n,-A,-P)
A	i, n, P	PMT	=PMT(i%,n,-P)
A	i, n, F	PMT	=PMT(i%,n,,-F)
A	i, n, P, F	PMT	=PMT(i%,n,-P,-F)
i	n, P, A	RATE	=RATE(n,A,-P)
i	n, P, F	RATE	=RATE(n,,-P,F)
i	n, A, F	RATE	=RATE(n,A,-F)
i	n, P, A, F	RATE	=RATE(n,A,-P,F)
n	i, P, A	NPER	=NPER(i%,A,-P)
n	i, P, F	NPER	=NPER(i%,,-P,F)
n	i, A, F	NPER	=NPER(i%,-A,,F)
n	i, P, A, F	NPER	=NPER(i%,A,-P,F)
i_{eff}	r, m	EFFECT	=EFFECT(r%,m)

Continuous Compounding

- In businesses and governments, transactions occur every year, every month, every day, every hour, every minute, every second!
- In multi-national corporations, for example, money is "put to work" immediately.
- Via electronic transfers, money is invested around the world continuously.
- Explicit consideration of "around the clock" and "around the world" money management motivates the use of continuous compounding.

Continuous Compounding

Let m denote the number of compounding periods in a year, r denote the nominal annual interest rate, and n denote the number of years. The single payment compound amount factor is given by $(1+r/m)^{mn}$

Letting the number of compounding periods in a year become infinitely large, the single payment compound amount factor reduces to

$$\lim_{m \to \infty} (1 + r/m)^{mn} = e^{rn}$$

Hence,

$$F = Pe^{rn}$$

$$F = P(F|P\ r\%, n)_{\infty}$$

$P = Fe^{-rn}$ **and**

$$P = F(P|F\ r\%, n)_{\infty}$$

Continuous Compounding

Recall, in discussing effective interest rates we claimed the effective interest rate for 12% compounded continuously was 12.7496851579%.

The effective interest rate under continuous compounding is given by

$$i_{eff} = e^r - 1 \quad \text{or}$$

$$i_{eff} = (F|P\ r\%,1)_{\infty} - 1$$

Therefore, the effective interest rate for 12% compounded continuously is equal to $e^{0.12}-1$, or 12.7496851579%.

Example 2.A.1

If $2,000 is invested in a fund that pays interest at a rate of 12% *compounded continuously*, after 5 years how much will be in the fund?

$$F = P(F|P\ 12\%,5)_{\infty} \quad = \$2,000(1.82212) = \$3,644.24$$

Example 2.A.2

If \$1,000 is deposited annually in an account that pays interest at a rate of 12% *compounded continuously*, after the 10^{th} deposit how much will be in the fund?

F = \$1,000(F|A 12%,10)$_{\infty}$

F = \$1,000(18.19744)

F = \$18,197.44

F =FV(exp(0.12)-1,10,-1000)

F = \$18,197.44

Example 2.A.2

$1,000 is deposited annually in an account that pays interest at a rate of 12% *compounded continuously*. **What is the present worth of the 10-year investment?**

P = $1,000(P|A 12%,10) $_\infty$

P = $1,000(5.48097)

P = $5,480.97

P =PV(exp(0.12)-1,10,-1000)

P = $5,480.97

Continuous Compounding

For the case of discrete cash flows, the continuous compounding equivalents for the discounted cash flow formulas can be obtained by substituting $(e^r\text{-}1)$ for i.

$(P \mid F\ \ r\%, n)_\infty = e^{-rn}$

$(F \mid P\ \ r\%, n)_\infty = e^{rn}$

$(F \mid A\ \ r\%, n)_\infty = (e^{rn}-1)/(e^r-1)$

$(A \mid F\ \ r\%, n)_\infty = (e^r-1)/(e^{rn}-1)$

$(P \mid A\ \ r\%, n)_\infty = (e^{rn}-1)/[e^{rn}(e^r-1)]$

$(A \mid P\ \ r\%, n)_\infty = e^{rn}(e^r-1)/(e^{rn}-1)$

See Table 2.A.1

Summary of continuous compounding interest factors for discrete flows.

To Find	Given	Factor	Symbol
P	F	e^{-rn}	$(P\|F\ r,n)_\infty$
F	P	e^{rn}	$(F\|P\ r,n)_\infty$
F	A	$\dfrac{e^{rn}-1}{e^r-1}$	$(F\|A\ r,n)_\infty$
A	F	$\dfrac{e^r-1}{e^{rn}-1}$	$(A\|F\ r,n)_\infty$
P	A	$\dfrac{e^{rn}-1}{e^{rn}(e^r-1)}$	$(P\|A\ r,n)_\infty$
A	P	$\dfrac{e^{rn}(e^r-1)}{e^{rn}-1}$	$(A\|P\ r,n)_\infty$
P	G	$\dfrac{e^{rn}-1-n(e^r-1)}{e^{rn}(e^r-1)^2}$	$(P\|G\ r,n)_\infty$
A	G	$\dfrac{1}{e^r-1}-\dfrac{n}{e^{rn}-1}$	$(A\|G\ r,n)_\infty$
P	A_1, c	$\dfrac{1-e^{(c-r)n}}{e^r-e^c}$	$(P\|A_1\ r,c,n)_\infty^*$
F	A_1, c	$\dfrac{e^{rn}-e^{cn}}{e^r-e^c}$	$(F\|A_1\ r,c,n)_\infty^*$

* $r \neq c$.

Example 2.A.3

Annual bonuses are deposited in a savings account that pays 8 percent *compounded continuously*. The size of the bonus *increases* at a rate of 10% *compounded continuously*; the initial bonus was \$500. How much will be in the account immediately after the 10th deposit?

$A_1 = \$500,\ r = 8\%,\ c = 10\%,\ n = 10,\ F = ?$

$F = \$500(F|A_1\ 8\%,10\%,10)^{\infty}$

$F = \$500(22.51619)$

$F = \$11,258.09$

Continuous Compounding, Continuous Flow

Not only does compounding of money occur "continuously," but also expenditures occur by the hour, minute, and second! Instead of cash flows for labor, material, energy, etc. occurring at the end of the year, they occur throughout the year, even daily or hourly. Hence, funds also flow continuously.

In the text, we show that the annual discrete cash flow (A) equivalent to an annual continuous cash flow ($\bar{A}$), based on an annual nominal interest rate of r, is given by

$$A = \bar{A}(e^r - 1)/r$$

Summary of continuous compounding interest factors for continuous flows.

Find	Given	Factor	Symbol
P	$\bar{A}$	$(e^{rn}-1)/(re^{rn})$	$(P\|\bar{A}\ r\%, n)$
$\bar{A}$	P	$re^{rn}/(e^{rn}-1)$	$(\bar{A}\|P\ r\%, n)$
F	$\bar{A}$	$(e^{rn}-1)/r$	$(F\|\bar{A}\ r\%, n)$
$\bar{A}$	F	$r/(e^{rn}-1)$	$(\bar{A}\|F\ r\%, n)$

Example 2.A.4

Determine the present worth equivalent of *a uniform series of continuous cash flows* totaling \$10,000/yr for 10 years when the interest rate is 20% *compounded continuously*.

P = \$10,000(P|Ā 20%,10)

P = \$10,000(4.32332)

P = \$43,233.20

P =PV(exp(0.2)-1,10,-10000*(exp(0.2)-1)/0.2)

P = \$43,233.24

Example 2.A.4

Determine the future worth equivalent of *a uniform series of continuous cash flows* totaling \$10,000/yr for 10 years when the interest rate is 20% per year *compounded continuously*.

F = \$10,000(F|Ā 20%,10)

F = \$10,000(31.94528)

F = \$319,452.80

F =FV(exp(0.2)-1,10,-10000*(exp(0.2)-1)/0.2)

F = \$319,452.80