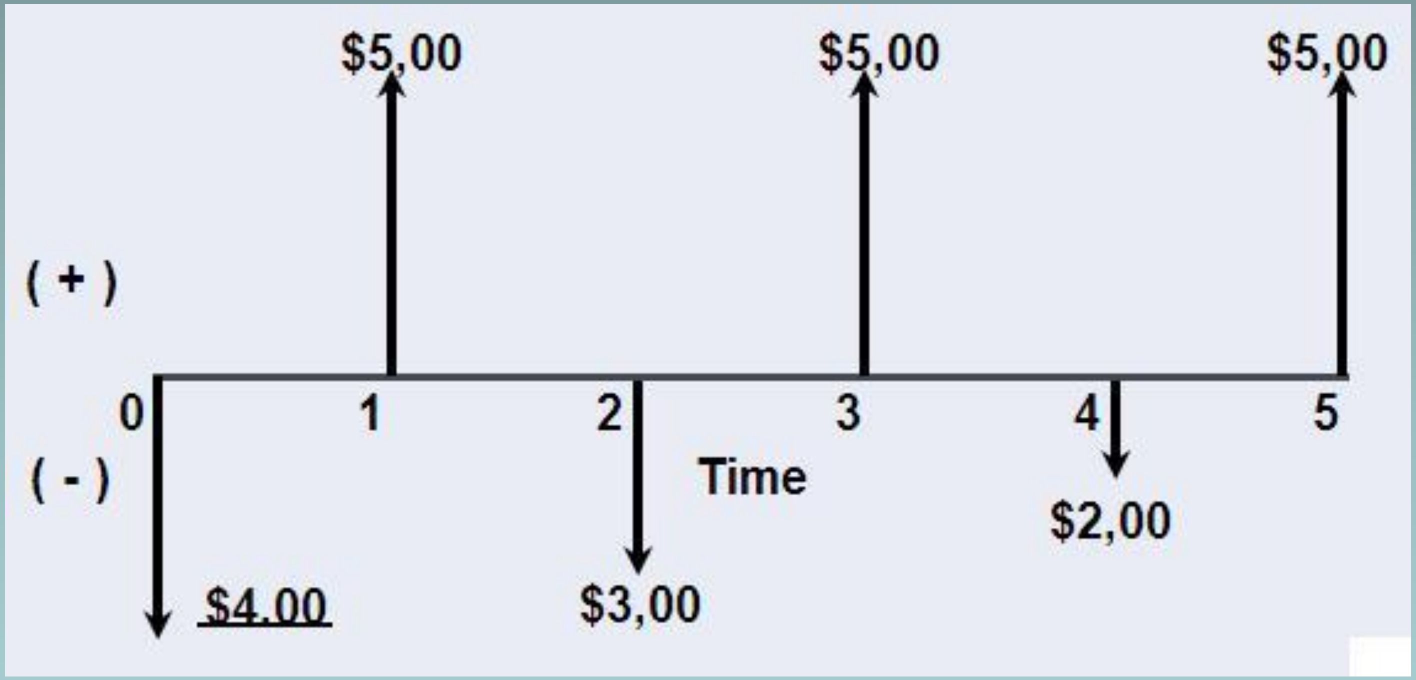
Chapter 2

Time Value of Money

(TVOM)

Cash Flow Diagrams



(EOY)

Example 2.1

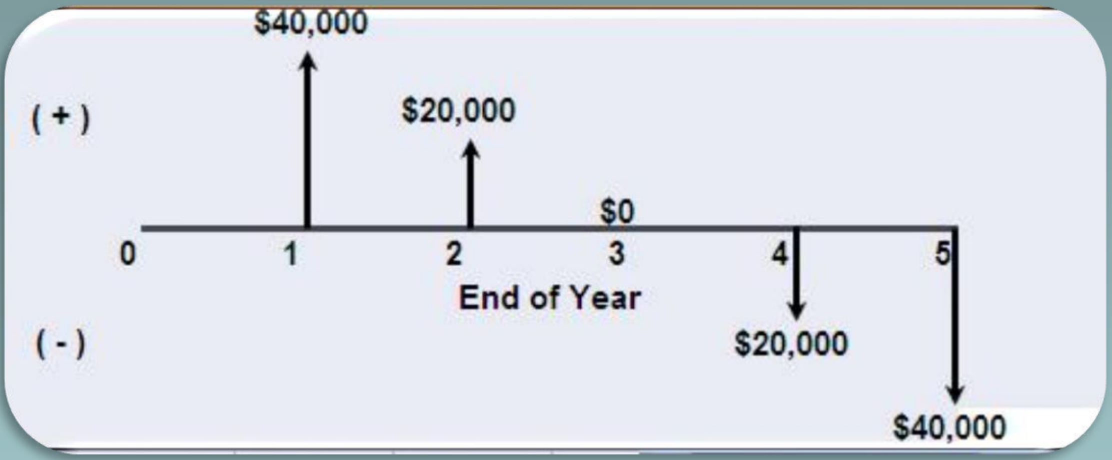
Cash Flow Profiles for Two Investment Alternatives

End of Year (EOY)	CF(A)	CF(B)	CF(B-A)
0	-$100,000	-$100,000	$0
1	$10,000	$50,000	$40,000
2	$20,000	$40,000	$20,000
3	$30,000	$30,000	$0
4	$40,000	$20,000	-$20,000
5	$50,000	$10,000	-$40,000
Sum	$50,000	$50,000	$0

Although the two investment alternatives have the same "bottom line," there are obvious differences. Which would you prefer, A or B? Why?

Principle #7

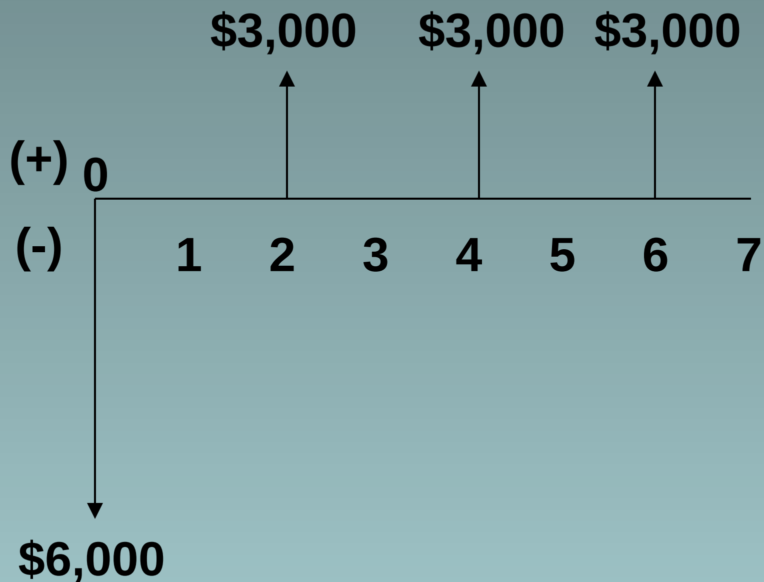
Consider only differences in cash flows among investment alternatives



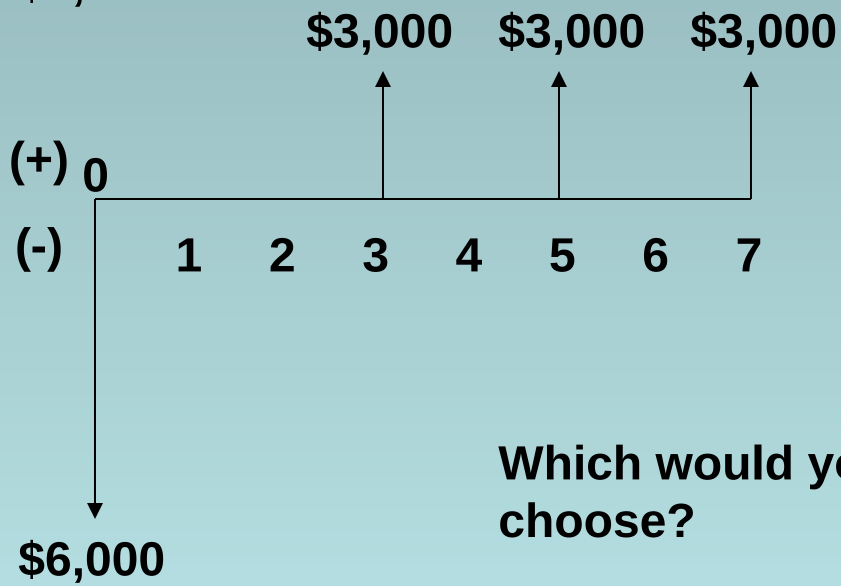
Inv. B – Inv. A

Example 2.2

Alternative C

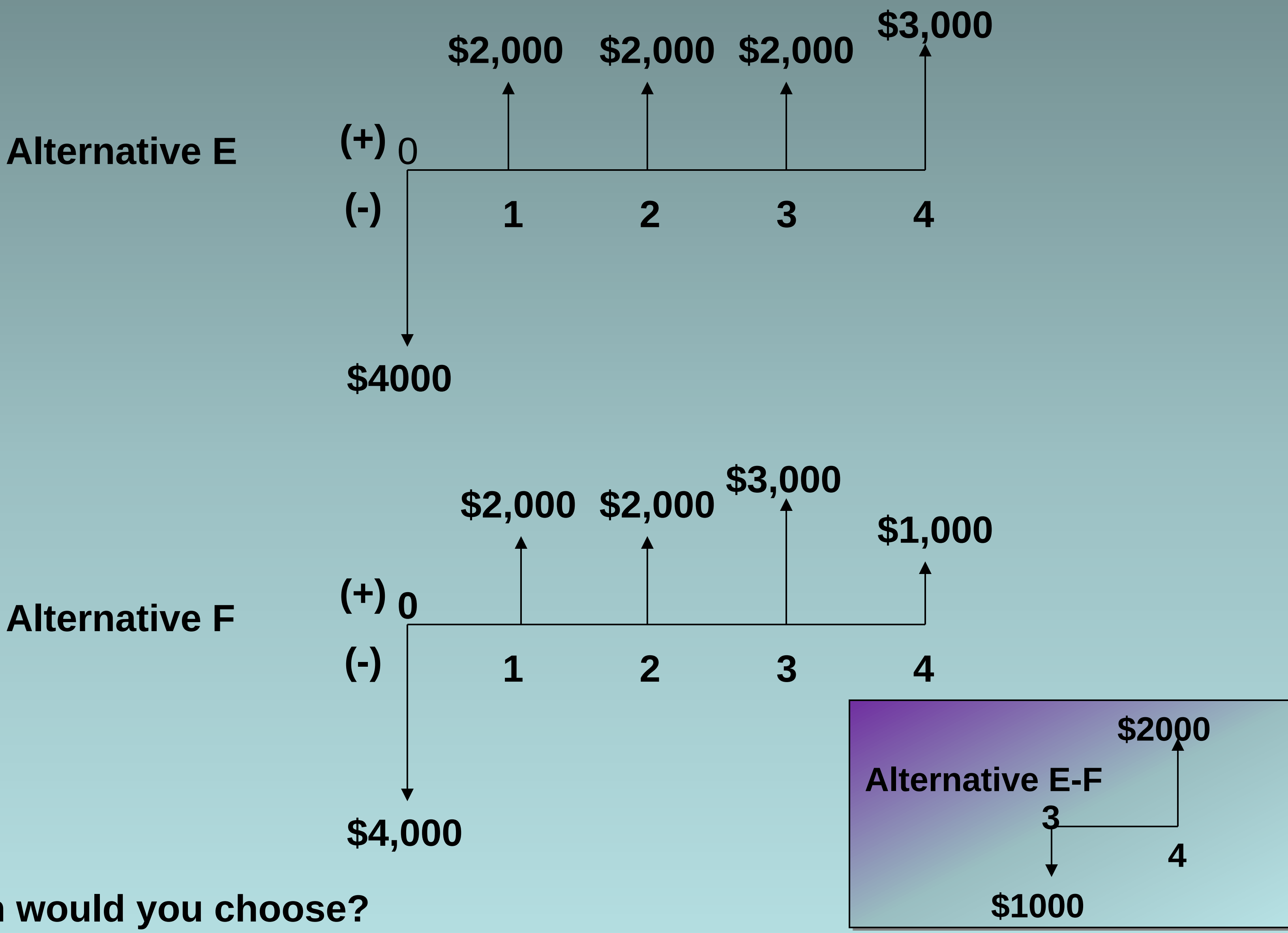


Alternative D



Which would you choose?

Example 2.3



Which would you choose?

- **Simple interest calculation:**

$$F_n = P(1 + in)$$

- **Compound Interest Calculation:**

$$F_n = F_{n-1}(1 + i)$$

  - **Where**
    - **P = present value of single sum of money**
    - **F_n = accumulated value of P over n periods**
    - **i = interest rate per period**
    - **n = number of periods**

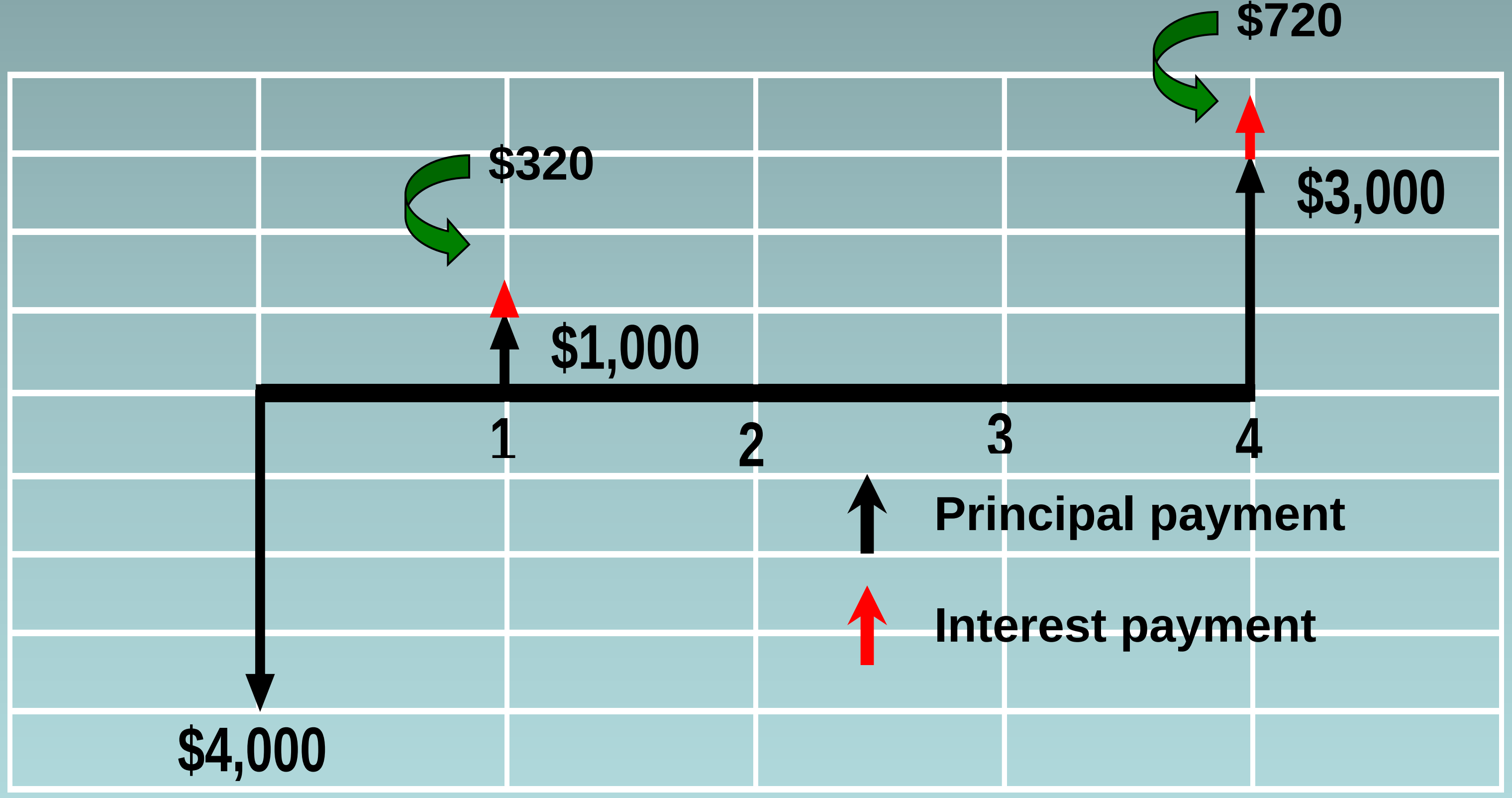
Example 2.7: simple interest calculation

Robert borrows $4,000 from Susan and agrees to pay $1,000 plus accrued interest at the end of the first year and $3,000 plus accrued interest at the end of the fourth year. What should be the size of the payments if 8% simple interest is used?

Solution

- **1^{st} payment = $1,000 + 0.08($4,000)**
  = $1,320
- **2^{nd} payment = $3,000 + 0.08($3,000)(3)**
  = $3,720

Simple Interest Cash Flow Diagram



RULES Discounting Cash Flow

1. Money has ***time value***!
2. Cash flows cannot be added ***unless they occur*** at the same point(s) in time.
3. ***Multiply*** a cash flow by ***(1+i)*** to move it ***forward one time unit***.
4. ***Divide*** a cash flow by ***(1+i)*** to move it ***backward one time unit***.

Example 2.8

(Lender's Perspective) Value of $10,000 Investment Growing @ 10% per year

Start of Year	Value of Investment	Interest Earned	End of Year	Value of Investment
1	$10,000.00	$1,000.00	1	$11,000.00
2	$11,000.00	$1,100.00	2	$12,100.00
3	$12,100.00	$1,210.00	3	$13,310.00
4	$13,310.00	$1,331.00	4	$14,641.00
5	$14,641.00	$1,464.10	5	$16,105.10

Compounding of Money

Beginning of Period	Amount Owed	Interest Earned	End of Period	Amount Owed
1	P	Pi	1	$P(1+i)$
2	$P(1+i)$	$P(1+i)i$	2	$P(1+i)^2$
3	$P(1+i)^2$	$P(1+i)^2i$	3	$P(1+i)^3$
4	$P(1+i)^3$	$P(1+i)^3i$	4	$P(1+i)^4$
5	$P(1+i)^4$	$P(1+i)^4i$	5	$P(1+i)^5$
. . .	. . .	. . .	. . .	. . .
n-1	$P(1+i)^{n-2}$	$P(1+i)^{n-2}i$	n-1	$P(1+i)^{n-1}$
n	$P(1+i)^{n-1}$	$P(1+i)^{n-1}i$	n	$P(1+i)^n$

Discounted Cash Flow Formulas

$$F = P(1+i)^{n} \qquad (2.8)$$

$F = P\,(F|P\ i\%, n)$

Vertical line means "given"

$$P = F(1+i)^{-n} \qquad (2.9)$$

$P = F\,(P|F\ i\%, n)$

Excel® DCF Worksheet Functions

$$F = P(1+i)^n \quad (2.1)$$

$$F = P(F|P\ i\%, n)$$

F =FV(i%,n,,-P)

$$P = F(1+i)^{-n} \quad (2.3)$$

$$P = F(P|F\ i\%, n)$$

P =PV(i%,n,,-F)

$F = P(1 + i)^n$

$F = P(F|P\ i\%,n)$ factor

F =FV(i%,n,,-P)

single sum, future worth

$P = F(1 + i)^{-n}$

$P = F(P|F\ i\%,n)$ factor

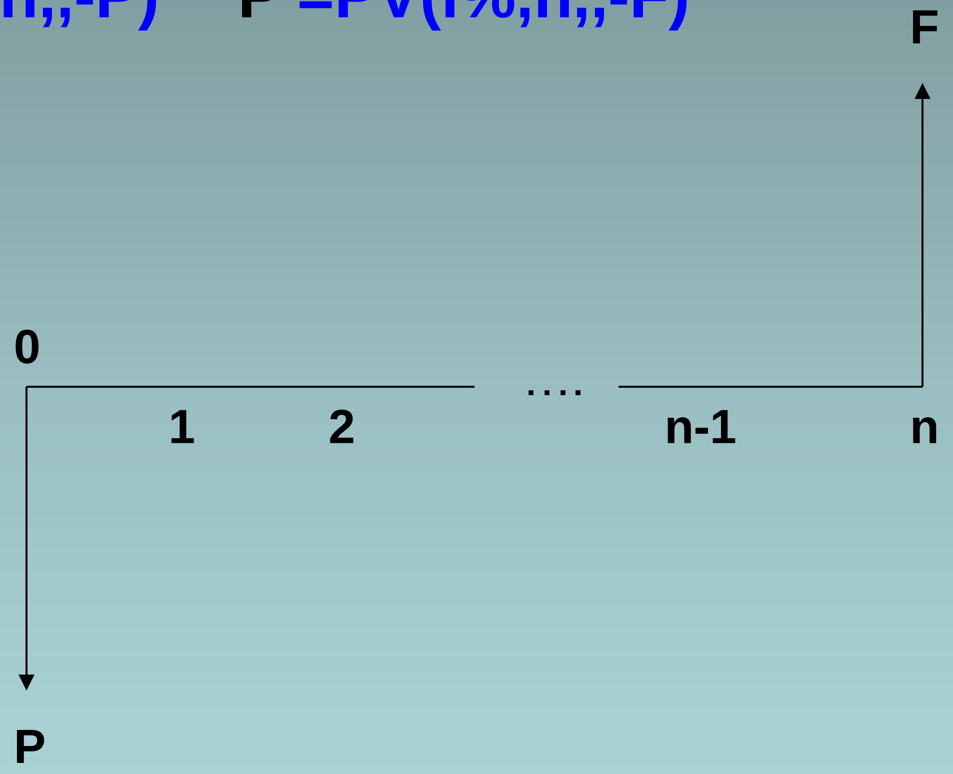
P =PV(i%,n,,-F)

single sum, present worth

$F = P(1+i)^n$ $P = F(1+i)^{-n}$

$F = P(F|P\ i\%, n)$ $P = F(P|F\ i\%, n)$

F =FV(i%,n,,-P) P =PV(i%,n,,-F)



P occurs n periods before F (F occurs n periods after P)

Relationships among P, F, and A

P occurs at the same time as A_0, i.e., at $t = 0$

F occurs at the same time as A_n, i.e., at $t = n$

Example 2.9

Dia St. John borrows $1,000 at 12% compounded annually. The loan is to be repaid after 5 years. How much must she repay in 5 years?

F = P(F|P i, n)

F = $1,000(F|P 12%,5)

F = $1,000(1.12)^5

F = $1,000(1.76234)

F = $1,762.34

F =FV(12%,5,,-1000)

F = $1,762.34

Example 2.11

How much must you deposit, today, in order to accumulate $10,000 in 4 years, if you earn 5% compounded annually on your investment?

P = F(P|F i, n)

P = $10,000(P|F 5%,4)

P = $10,000(1.05)$^{-4}$

P = $10,000(0.82270)

P = $8,227.00

P =PV(5%,4,,-10000)

P = $8227.02

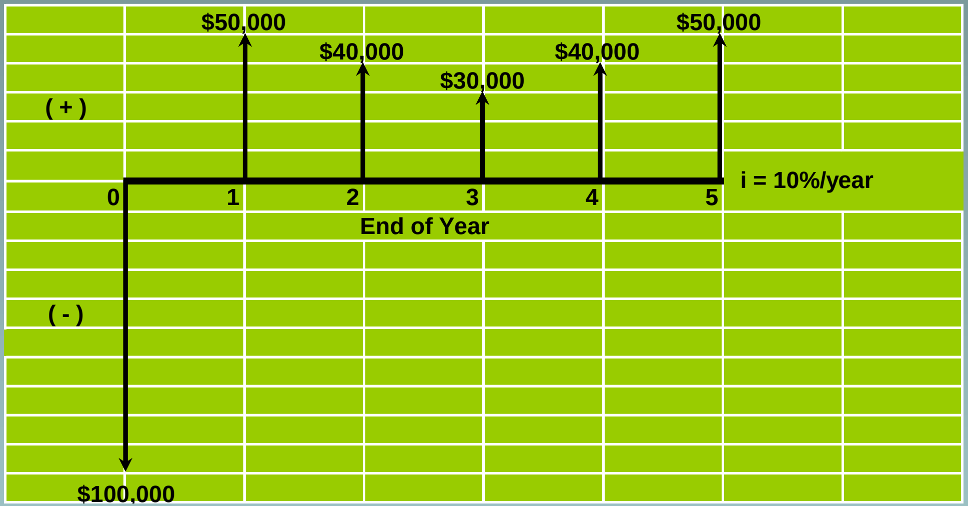
Computing the Present Worth of Multiple Cash flows

$$P = \sum_{t=0}^{n} A_t (1+i)^{-t} \qquad (2.12)$$

$$P = \sum_{t=0}^{n} A_t (P \mid F\ i\%,\ t) \qquad (2.13)$$

Example 2.12

Determine the present worth equivalent of the CFD shown below, using an interest rate of 10% compounded annually.

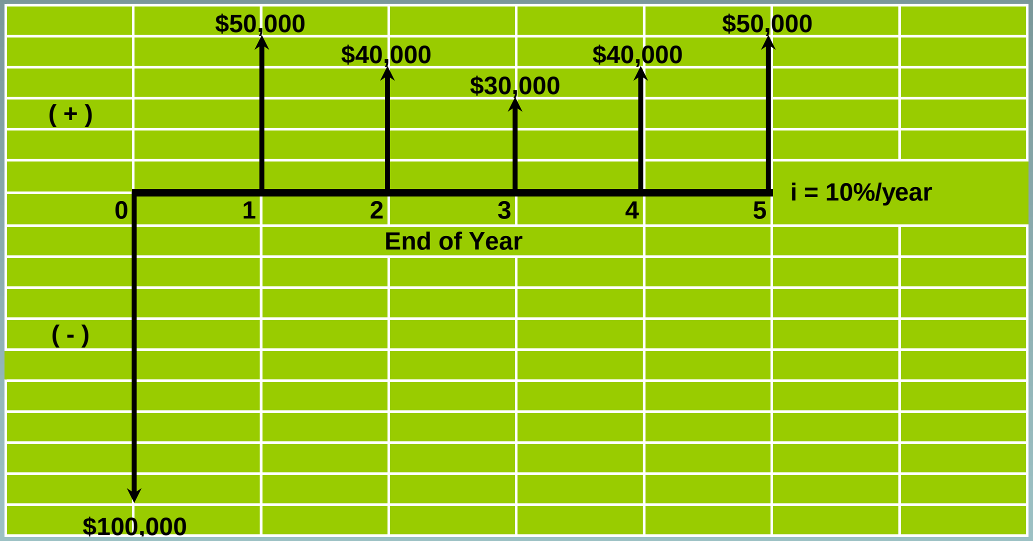


End of Year (n)	Cash Flow (CF)	(*P* |*F* 10%,n)	Present Worth	PV(10%,n,,-CF)	(*F* |*P* 10%,5-n)	Future Worth	FV(10%,5-n,,-CF)
0	-$100,000	1.00000	-$100,000.00	-$100,000.00	1.61051	-$161,051.00	-$161,051.00
1	$50,000	0.90909	$45,454.50	$45,454.55	1.46410	$73,205.00	$73,205.00
2	$40,000	0.82645	$33,058.00	$33,057.85	1.33100	$53,240.00	$53,240.00
3	$30,000	0.75131	$22,539.30	$22,539.44	1.21000	$36,300.00	$36,300.00
4	$40,000	0.68301	$27,320.40	$27,320.54	1.10000	$44,000.00	$44,000.00
5	$50,000	0.62092	$31,046.00	$31,046.07	1.00000	$50,000.00	$50,000.00
SUM			$59,418.20	$59,418.45		$95,694.00	$95,694.00

P =NPV(10%,50000,40000,30000,40000,50000)-100000
= $59,418.45

Example 2.15

Determine the future worth equivalent of the CFD shown below, using an interest rate of 10% compounded annually.



End of Year (n)	Cash Flow (CF)	(*P* |*F* 10%,n)	Present Worth	PV(10%,n,,-CF)	(*F* |*P* 10%,5-n)	Future Worth	FV(10%,5-n,,-CF)
0	-$100,000	1.00000	-$100,000.00	-$100,000.00	1.61051	-$161,051.00	-$161,051.00
1	$50,000	0.90909	$45,454.50	$45,454.55	1.46410	$73,205.00	$73,205.00
2	$40,000	0.82645	$33,058.00	$33,057.85	1.33100	$53,240.00	$53,240.00
3	$30,000	0.75131	$22,539.30	$22,539.44	1.21000	$36,300.00	$36,300.00
4	$40,000	0.68301	$27,320.40	$27,320.54	1.10000	$44,000.00	$44,000.00
5	$50,000	0.62092	$31,046.00	$31,046.07	1.00000	$50,000.00	$50,000.00
SUM			$59,418.20	$59,418.45		$95,694.00	$95,694.00

F =10000*FV(10%,5,,-NPV(10%,5,4,3,4,5)+10)

= $95,694.00

Examples 2.13 & 2.16

Determine the present worth equivalent of the following series of cash flows. Use an interest rate of 6% per interest period.

End of Period	Cash Flow
0	$0
1	$300
2	$0
3	-$300
4	$200
5	$0
6	$400
7	$0
8	$200

P = $300(P|F 6%,1)- $300(P|F 6%,3)+$200(P|F 6%,4)+$400(P|F 6%,6)
+$200(P|F 6%,8) = $597.02

P =NPV(6%,300,0,-300,200,0,400,0,200)

P =$597.02

Computing the Future worth of Multiple cash Flows

$$F = \sum_{t=1}^{n} A_t (1+i)^{n-t} \qquad (2.15)$$

$$F = \sum_{t=1}^{n} A_t (F \mid P \quad i\%, n-t) \qquad (2.16)$$

Examples 2.14 & 2.16

Determine the future worth equivalent of the following series of cash flows. Use an interest rate of 6% per interest period.

End of Period	Cash Flow
0	$0
1	$300
2	$0
3	-$300
4	$200
5	$0
6	$400
7	$0
8	$200

F = $300(F|P 6%,7)-$300(F|P 6%,5)+ $200(F|P 6%,4)+$400(F|P 6%,2)+$200
F = $951.59

F =FV(6%,8,,-NPV(6%,300,0,-300,200,0,400,0,200))
F =$951.56

(The 3¢ difference in the answers is due to round-off error in the tables in Appendix A.)

End of Year (n)	Cash Flow (CF)	
0	$0	
1	$300	
2	$0	
3	-$300	
4	$200	
5	$0	
6	$400	
7	$0	
8	$200	
P =	$597.02	=NPV(6%,C4:C11)
F =	$951.56	=FV(6%,8,,-C12)

Some Common Cash Flow Series

- **Uniform Series**

$$A_t = A \qquad t = 1,\ldots,n$$

- **Gradient Series**

$$A_t = 0 \qquad t = 1$$

$$= A_{t-1}+G \qquad t = 2,\ldots,n$$

$$= (t-1)G \qquad t = 1,\ldots,n$$

- **Geometric Series**

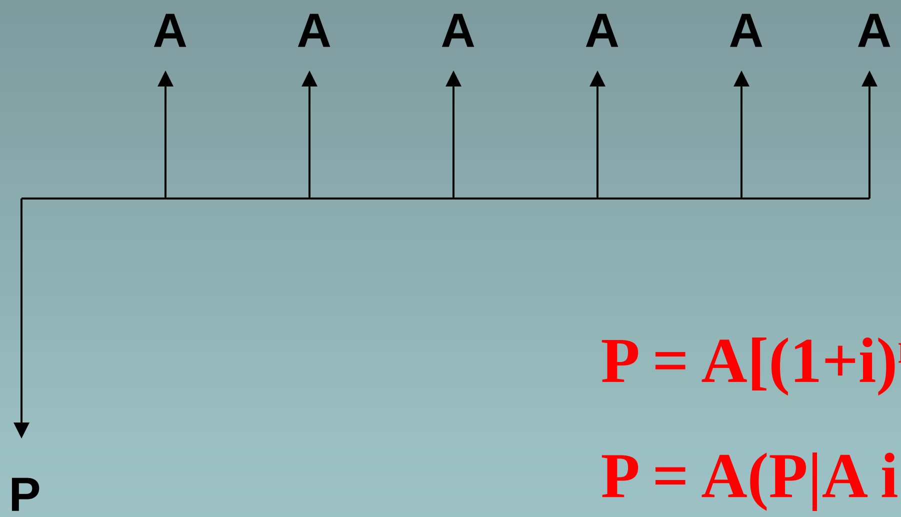
$$A_t = A \qquad t = 1$$

$$= A_{t-1}(1+j) \qquad t = 2,\ldots,n$$

$$= A_1(1+j)^{t-1} \qquad t = 1,\ldots,n$$

Uniform Series

DCF Uniform Series Formulas



P occurs 1 period

before first A

$$P = A[(1+i)^n-1]/[i(1+i)^n]$$

$$P = A(P|A\ i\%,n)$$

$$P = [\ =PV(i\%,n,-A)\ ]$$

$$A = Pi(1+i)^n/[(1+i)^n-1]$$

$$A = P(A|P\ i\%,n)$$

$$A = PMT(i\%,n,-P)$$

$$F = A[(1+i)^n - 1]/i$$

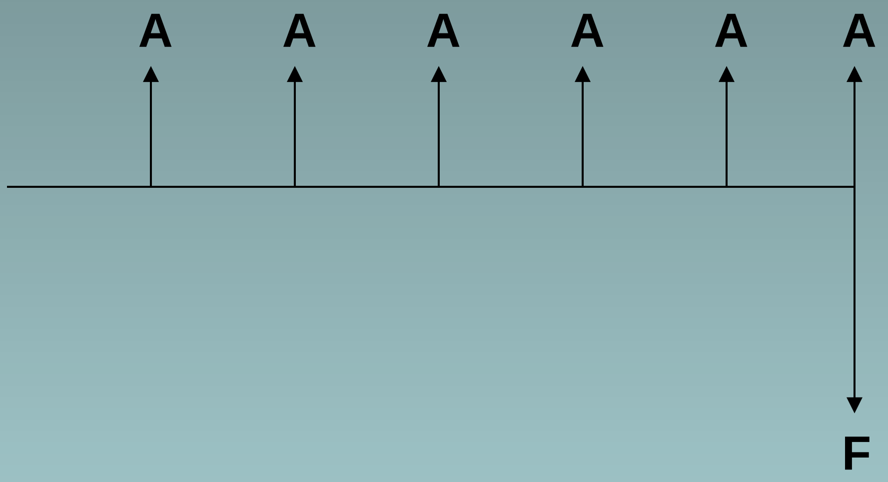
$$F = A(F|A\ i\%,n)$$

$$F = FV(i\%,n,-A)$$

$$A = Fi/[(1+i)^n - 1]$$

$$A = F(A|F\ i\%,n)$$

$$A = PMT(i\%,n,,-F)$$



F occurs at the same time as last A

$$P = A(P|A\ i\%,n) = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad (2.22)$$

$$A = P(A|P\ i\%,n) = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] \quad (2.25)$$

P occurs one period before the first A

$$F = A(F|A\ i\%,n) = A\left[\frac{(1+i)^n - 1}{i}\right] \quad (2.28)$$

$$A = F(A|F\ i\%,n) = F\left[\frac{i}{(1+i)^n - 1}\right] \quad (2.30)$$

F occurs at the same time as the last A

Example 2. 17

Troy Long deposits a single sum of money in a savings account that pays 5% compounded annually. How much must he deposit in order to withdraw $2,000/yr for 5 years, with the first withdrawal *occurring 1 year after* the deposit?

P = $2000(P|A 5%,5)

P = $2000(4.32948) = $8658.96

P =PV(5%,5,-2000)

P = $8658.95

Example 2. 18

Troy Long deposits a single sum of money in a savings account that pays 5% compounded annually. How much must he deposit in order to withdraw $2,000/yr for 5 years, *with the first withdrawal occurring 3 years after the deposit?*

P = $2000(P|A 5%,5)(P|F 5%,2)

P = $2000(4.32948)(0.90703) = $7853.94

P =PV(5%,2,,-PV(5%,5,-2000))

P = $7853.93

Example 2. 19

Rachel Townsley invests $10,000 in a fund that pays 8% compounded annually. If she makes 10 equal annual withdrawals from the fund, how much can she withdraw if the first withdrawal occurs *1 year after* her investment?

A = $10,000(A|P 8%,10)
A = $10,000(0.14903) = $1490.30

A =PMT(8%,10,-10000)
A = $1490.29

Example 2.22

Suppose Rachel delays the first withdrawal for 2 years. How much can be withdrawn each of the 10 years?

A = \$10,000(F|P 8%,2)(A|P 8%,10)

A = \$10,000(1.16640)(0.14903)

A = \$1738.29

A =PMT(8%,10-FV(8%,2,,-10000))

A = \$1738.29

Example 2.20

A firm borrows $2,000,000 at 12% annual interest and pays it back with 10 equal annual payments. What is the payment?

A = $2,000,000(A|P 12%,10)

A = $2,000,000(0.17698)

A = $353,960

A =PMT(12%,10,-2000000)

A = $353,968.33

Example 2.21

Suppose the firm pays back the loan over 15 years in order to obtain a 10% interest rate. What would be the size of the annual payment?

A = $2,000,000(A|P 10%,15)

A = $2,000,000(0.13147)

A = $262,940

A =PMT(10%,15,-2000000)

A = $262,947.55

Extending the loan period 5 years reduced the payment by $91,020.78

Example 2. 23

Luis Jimenez deposits $1,000/yr in a savings account that pays 6% compounded annually. How much will be in the account *immediately after* his 30th deposit?

F = 1000(F|A 6%,30)

F = $1000(79.05819) = $79,058.19

F =FV(6%,30,-1000)

A = $78,058.19

Example 2. 24

Andrew Brewer invests $5,000/yr and earns 6% compounded annually. How much will he have in his investment portfolio after 15 yrs? 20 yrs? 25 yrs? 30 yrs? (What if he earns 3%/yr?)

F = $5000(F|A 6%,15) = $5000(23.27597) = $116,379.85

F = $5000(F|A 6%,20) = $5000(36.78559) = $183,927.95

F = $5000(F|A 6%,25) = $5000(54.86451) = $274,322.55

F = $5000(F|A 6%,30) = $5000(79.05819) = $395,290.95

F = $5000(F|A 3%,15) = $5000(18.59891) = $92,994.55

F = $5000(F|A 3%,20) = $5000(26.87037) = $134,351.85

F = $5000(F|A 3%,25) = $5000(36.45926) = $182,296.30

F = $5000(F|A 3%,30) = $5000(47.57542) = $237,877.10

Example 2.25

If Coby Durham earns 7% on his investments, how much must he invest annually in order to accumulate $1,500,000 in 25 years?

A = $1,500,000(A|F 7%,25)

A = $1,500,000(0.01581)

A = $23,715

A =PMT(7%,25,,-1500000)

A = $23,715.78

Example 2.26

If Crystal Wilson earns 10% on her investments, how much must she invest annually in order to accumulate $1,000,000 in 40 years?

A = $1,000,000(A|F 10%,40)

A = $1,000,000(0.0022594)

A = $2,259.40

A =PMT(10%,40,,-1000000)

A = $2,259.41

Example 2.27

$500,000 is spent for a SMP machine in order to reduce annual expenses by $92,500/yr. At the end of a 10-year planning horizon, the SMP machine is worth $50,000. Based on a 10% TVOM,

a) what single sum at t = 0 is equivalent to the SMP investment?

b) what single sum at t = 10 is equivalent to the SMP investment?

c) what uniform annual series over the 10-year period is equivalent to the SMP investment?

Solution:

P = -$500,000 + $92,500(P|A 10%,10) + $50,000(P|F 10%,10)

P = -$500,000 + $92,500(6.14457) + $50,000(0.38554) = $87,649.73

P =PV(10%,10,-92500,-50000)-500000 = $87,649.62

Example 2.27 (Solution)

F = -$500,000(F|P 10%,10) + $92,500(F|A 10%,10) + $50,000

F = -$500,000(2.59374) + $92,500(15.93742) + $50,000 = $227,341.40

F =FV(10%,10,-92500,500000)+50000 = $227,340.55

A = -$500,000(A|P 10%,10) + $92,500 + $50,000(A|F 10%,10)

A = -$500,000(0.16275) + $92,500 + $50,000(0.06275) = $14,262.50

A =PMT(10%,10,500000,-50000)+92500 = $14,264.57

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

uniform series, present worth factor

= *A*(*P*|*A i*%,*n*) **=PV(*i*%,*n*,-*A*)**

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

uniform series, capital recovery factor

= *P*(*A*|*P i*%,*n*) **=PMT(*i*%,*n*,-*P*)**

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

uniform series, future worth factor

= *A*(*F*|*A i*%,*n*) **=FV(*i*%,*n*,-*A*)**

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

uniform series, sinking fund factor

= *F*(*A*|*F i*%,*n*) **=PMT(*i*%,*n*,,-*F*)**