Appendix

C

Selected Topics of Mathematics

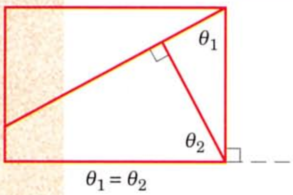
C/1 Introduction

Appendix C contains an abbreviated summary and reminder of selected topics in basic mathematics which find frequent use in mechanics. The relationships are cited without proof. The student of mechanics will have frequent occasion to use many of these relations, and he or she will be handicapped if they are not well in hand. Other topics not listed will also be needed from time to time.

As the reader reviews and applies mathematics, he or she should bear in mind that mechanics is an applied science descriptive of real bodies and actual motions. Therefore, the geometric and physical interpretation of the applicable mathematics should be kept clearly in mind during the development of theory and the formulation and solution of problems.

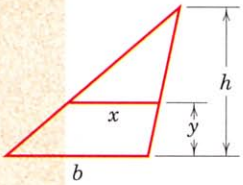
C/2 Plane Geometry

1. When two intersecting lines are, respectively, perpendicular to two other lines, the angles formed by the two pairs are equal.



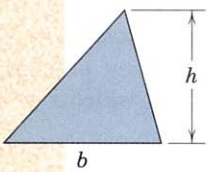
2. Similar triangles

$$\frac{x}{b} = \frac{h - y}{h}$$



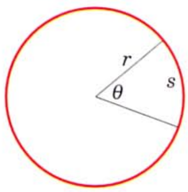
3. Any triangle

Area $= \frac{1}{2}bh$

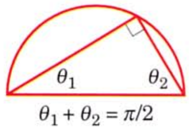


4. Circle

Circumference $= 2\pi r$
Area $= \pi r^2$
Arc length $s = r\theta$
Sector area $= \frac{1}{2}r^2\theta$

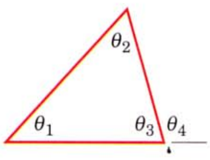


5. Every triangle inscribed within a semicircle is a right triangle.



6. Angles of a triangle

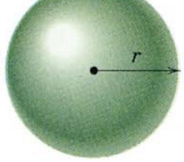
$\theta_1 + \theta_2 + \theta_3 = 180°$
$\theta_4 = \theta_1 + \theta_2$



C/3 Solid Geometry

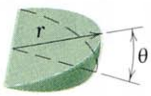
1. Sphere

Volume $= \frac{4}{3}\pi r^3$
Surface area $= 4\pi r^2$



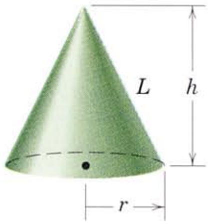
2. Spherical wedge

Volume $= \frac{2}{3}r^3\theta$



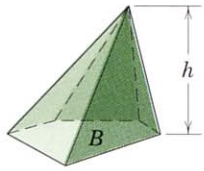
3. Right-circular cone

Volume $= \frac{1}{3}\pi r^2 h$
Lateral area $= \pi r L$
$L = \sqrt{r^2 + h^2}$



4. Any pyramid or cone

Volume $= \frac{1}{3}Bh$
where B = area of base



C/4 Algebra

1. Quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \; b^2 \geq 4ac \text{ for real roots}$$

2. Logarithms

$$b^x = y, \; x = \log_b y$$

Natural logarithms

$$b = e = 2.718\,282$$
$$e^x = y, \; x = \log_e y = \ln y$$

$$\log(ab) = \log a + \log b$$
$$\log(a/b) = \log a - \log b$$
$$\log(1/n) = -\log n$$
$$\log a^n = n \log a$$
$$\log 1 = 0$$
$$\log_{10} x = 0.4343 \ln x$$

3. Determinants

2nd order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

3rd order

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{aligned} &+a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ &-a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2 \end{aligned}$$

4. Cubic equation

$$x^3 = Ax + B$$

Let $p = A/3$, $q = B/2$.

Case I: $q^2 - p^3$ negative (three roots real and distinct)

$$\cos u = q/(p\sqrt{p}), \; 0 < u < 180°$$
$$x_1 = 2\sqrt{p} \cos(u/3)$$
$$x_2 = 2\sqrt{p} \cos(u/3 + 120°)$$
$$x_3 = 2\sqrt{p} \cos(u/3 + 240°)$$

Case II: $q^2 - p^3$ positive (one root real, two roots imaginary)

$$x_1 = (q + \sqrt{q^2 - p^3})^{1/3} + (q - \sqrt{q^2 - p^3})^{1/3}$$

Case III: $q^2 - p^3 = 0$ (three roots real, two roots equal)

$$x_1 = 2q^{1/3}, \; x_2 = x_3 = -q^{1/3}$$

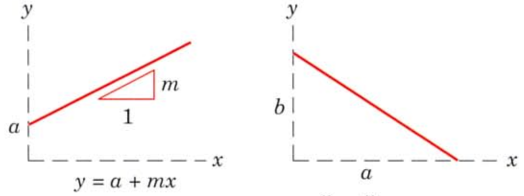
For general cubic equation

$$x^3 + ax^2 + bx + c = 0$$

Substitute $x = x_0 - a/3$ and get $x_0{}^3 = Ax_0 + B$. Then proceed as above to find values of x_0 from which $x = x_0 - a/3$.

C/5 ANALYTIC GEOMETRY

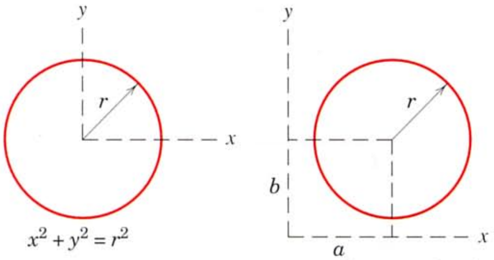
1. Straight line



$y = a + mx$

$\frac{x}{a} + \frac{y}{b} = 1$

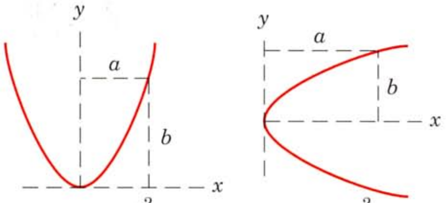
2. Circle



$x^2 + y^2 = r^2$

$(x - a)^2 + (y - b)^2 = r^2$

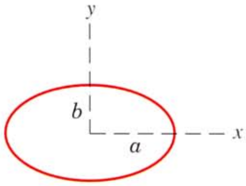
3. Parabola



$y = b\frac{x^2}{a^2}$

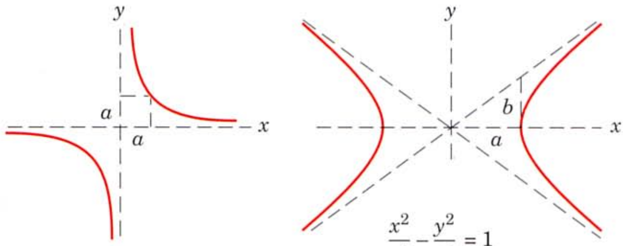
$x = a\frac{y^2}{b^2}$

4. Ellipse



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5. Hyperbola



$xy = a^2$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

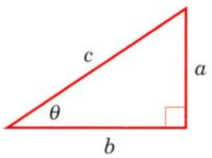
C/6 TRIGONOMETRY

1. Definitions

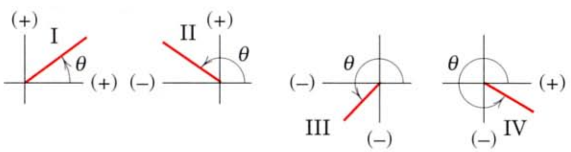
$\sin\theta = a/c \quad \csc\theta = c/a$

$\cos\theta = b/c \quad \sec\theta = c/b$

$\tan\theta = a/b \quad \cot\theta = b/a$



2. Signs in the four quadrants



	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\csc\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

3. Miscellaneous relations

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$1 + \tan^2 \theta = \sec^2 \theta$$
$$1 + \cot^2 \theta = \csc^2 \theta$$

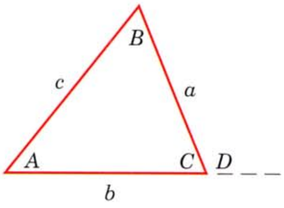
$$\sin \frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 - \cos \theta)}$$

$$\cos \frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 + \cos \theta)}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$$

4. Law of sines

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$



5. Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$c^2 = a^2 + b^2 + 2ab \cos D$$

C/7 Vector Operations

1. ***Notation.*** Vector quantities are printed in boldface type, and scalar quantities appear in lightface italic type. Thus, the vector quantity **V** has a scalar magnitude V. In longhand work vector quantities should always be consistently indicated by a symbol such as $\underline{V}$ or $\vec{V}$ to distinguish them from scalar quantities.

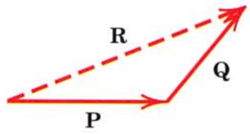
2. ***Addition***

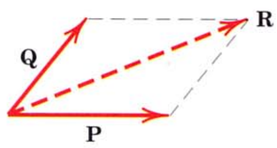
Triangle addition $\mathbf{P} + \mathbf{Q} = \mathbf{R}$

Parallelogram addition $\mathbf{P} + \mathbf{Q} = \mathbf{R}$

Commutative law $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$

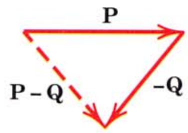
Associative law $\mathbf{P} + (\mathbf{Q} + \mathbf{R}) = (\mathbf{P} + \mathbf{Q}) + \mathbf{R}$





3. ***Subtraction***

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q})$$



4. ***Unit vectors*** **i**, **j**, **k**

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

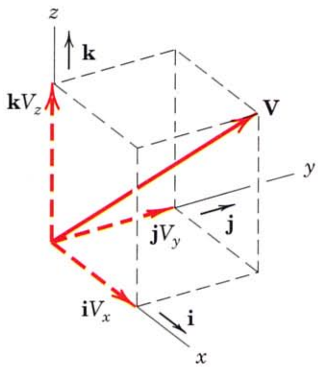
where $$|\mathbf{V}| = V = \sqrt{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}}$$

5. ***Direction cosines*** l, m, n are the cosines of the angles between **V** and the x-, y-, z-axes. Thus,

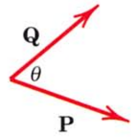
$$l = V_x/V \qquad m = V_y/V \qquad n = V_z/V$$

so that $$\mathbf{V} = V(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

and $$l^2 + m^2 + n^2 = 1$$

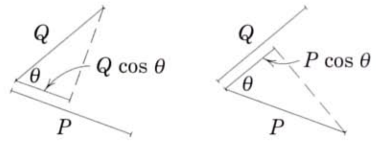


6. ***Dot or scalar product***



$$\mathbf{P}\cdot\mathbf{Q} = PQ\cos\theta$$

This product may be viewed as the magnitude of **P** multiplied by the component $Q\cos\theta$ of **Q** in the direction of **P**, or as the magnitude of **Q** multiplied by the component $P\cos\theta$ of **P** in the direction of **Q**.



Commutative law $\qquad \mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P}$

From the definition of the dot product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\cdot(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})\\ &= P_xQ_x + P_yQ_y + P_zQ_z\end{aligned}$$

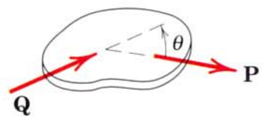
$$\mathbf{P}\cdot\mathbf{P} = P_x^{\,2} + P_y^{\,2} + P_z^{\,2}$$

It follows from the definition of the dot product that two vectors **P** and **Q** are perpendicular when their dot product vanishes, $\mathbf{P}\cdot\mathbf{Q} = 0$.

The angle θ between two vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ may be found from their dot product expression $\mathbf{P}_1\cdot\mathbf{P}_2 = P_1P_2\cos\theta$, which gives

$$\cos\theta = \frac{\mathbf{P}_1\cdot\mathbf{P}_2}{P_1P_2} = \frac{P_{1_x}P_{2_x} + P_{1_y}P_{2_y} + P_{1_z}P_{2_z}}{P_1P_2} = l_1l_2 + m_1m_2 + n_1n_2$$

where l, m, n stand for the respective direction cosines of the vectors. It is also observed that two vectors are perpendicular to each other when their direction cosines obey the relation $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

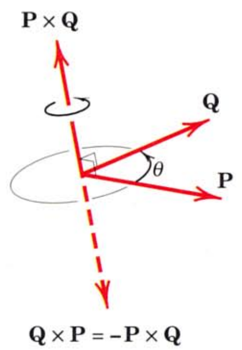


Distributive law $\qquad \mathbf{P}\cdot(\mathbf{Q} + \mathbf{R}) = \mathbf{P}\cdot\mathbf{Q} + \mathbf{P}\cdot\mathbf{R}$

7. ***Cross or vector product.*** The cross product $\mathbf{P}\times\mathbf{Q}$ of the two vectors **P** and **Q** is defined as a vector with a magnitude

$$|\mathbf{P}\times\mathbf{Q}| = PQ\sin\theta$$

and a direction specified by the right-hand rule as shown. Reversing the vector order and using the right-hand rule give $\mathbf{Q}\times\mathbf{P} = -\mathbf{P}\times\mathbf{Q}$.



Distributive law $\qquad \mathbf{P}\times(\mathbf{Q} + \mathbf{R}) = \mathbf{P}\times\mathbf{Q} + \mathbf{P}\times\mathbf{R}$

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k} \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i} \qquad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

With the aid of these identities and the distributive law, the vector product may be written

$$\begin{aligned}\mathbf{P} \times \mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}\end{aligned}$$

The cross product may also be expressed by the determinant

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

8. *Additional relations*

Triple scalar product $(\mathbf{P} \times \mathbf{Q})\cdot\mathbf{R} = \mathbf{R}\cdot(\mathbf{P} \times \mathbf{Q})$. The dot and cross may be interchanged as long as the order of the vectors is maintained. Parentheses are unnecessary since $\mathbf{P} \times (\mathbf{Q}\cdot\mathbf{R})$ is meaningless because a vector $\mathbf{P}$ cannot be crossed into a scalar $\mathbf{Q}\cdot\mathbf{R}$. Thus, the expression may be written

$$\mathbf{P} \times \mathbf{Q}\cdot\mathbf{R} = \mathbf{P}\cdot\mathbf{Q} \times \mathbf{R}$$

The triple scalar product has the determinant expansion

$$\mathbf{P} \times \mathbf{Q}\cdot\mathbf{R} = \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}$$

Triple vector product $(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R} = -\mathbf{R} \times (\mathbf{P} \times \mathbf{Q}) = \mathbf{R} \times (\mathbf{Q} \times \mathbf{P})$. Here we note that the parentheses must be used since an expression $\mathbf{P} \times \mathbf{Q} \times \mathbf{R}$ would be ambiguous because it would not identify the vector to be crossed. It may be shown that the triple vector product is equivalent to

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R} = \mathbf{R}\cdot\mathbf{P}\mathbf{Q} - \mathbf{R}\cdot\mathbf{Q}\mathbf{P}$$

or

$$\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = \mathbf{P}\cdot\mathbf{R}\mathbf{Q} - \mathbf{P}\cdot\mathbf{Q}\mathbf{R}$$

The first term in the first expression, for example, is the dot product $\mathbf{R}\cdot\mathbf{P}$, a scalar, multiplied by the vector $\mathbf{Q}$.

9. *Derivatives of vectors* obey the same rules as they do for scalars.

$$\frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{P}_x\mathbf{i} + \dot{P}_y\mathbf{j} + \dot{P}_z\mathbf{k}$$

$$\frac{d(\mathbf{P}u)}{dt} = \mathbf{P}\dot{u} + \dot{\mathbf{P}}u$$

$$\frac{d(\mathbf{P}\cdot\mathbf{Q})}{dt} = \mathbf{P}\cdot\dot{\mathbf{Q}} + \dot{\mathbf{P}}\cdot\mathbf{Q}$$

$$\frac{d(\mathbf{P} \times \mathbf{Q})}{dt} = \mathbf{P} \times \dot{\mathbf{Q}} + \dot{\mathbf{P}} \times \mathbf{Q}$$

10. ***Integration of vectors.*** If $\mathbf{V}$ is a function of x, y, and z and an element of volume is $d\tau = dx\, dy\, dz$, the integral of $\mathbf{V}$ over the volume may be written as the vector sum of the three integrals of its components. Thus,

$$\int \mathbf{V}\, d\tau = \mathbf{i} \int V_x\, d\tau + \mathbf{j} \int V_y\, d\tau + \mathbf{k} \int V_z\, d\tau$$

C/8 Series

(Expression in brackets following series indicates range of convergence.)

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad [x^2 < 1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad [x^2 < \infty]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad [x^2 < \infty]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad [x^2 < \infty]$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad [x^2 < \infty]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\text{where } a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

[Fourier expansion for $-l < x < l$]

C/9 Derivatives

$$\frac{dx^n}{dx} = nx^{n-1}, \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\lim_{\Delta x \to 0} \sin \Delta x = \sin dx = \tan dx = dx$$

$$\lim_{\Delta x \to 0} \cos \Delta x = \cos dx = 1$$

$$\frac{d \sin x}{dx} = \cos x, \qquad \frac{d \cos x}{dx} = -\sin x, \qquad \frac{d \tan x}{dx} = \sec^2 x$$

$$\frac{d \sinh x}{dx} = \cosh x, \qquad \frac{d \cosh x}{dx} = \sinh x, \qquad \frac{d \tanh x}{dx} = \operatorname{sech}^2 x$$

C/10 Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{x} = \ln x$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}$$

$$\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx-2a)\sqrt{(a+bx)^3}$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2}{105b^3}(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$$

$$\int \frac{\sqrt{a+x}}{\sqrt{b-x}}\,dx = -\sqrt{a+x}\sqrt{b-x} + (a+b)\sin^{-1}\sqrt{\frac{a+x}{a+b}}$$

$$\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}[a+bx-a\ln(a+bx)]$$

$$\int \frac{x\,dx}{(a+bx)^n} = \frac{(a+bx)^{1-n}}{b^2}\left(\frac{a+bx}{2-n} - \frac{a}{1-n}\right)$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} \quad \text{or} \quad \frac{1}{\sqrt{-ab}}\tanh^{-1}\frac{x\sqrt{-ab}}{a}$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(a+bx^2)$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2\ln(x+\sqrt{x^2 \pm a^2})]$$

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right)$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right)$$

$$\int x^3\sqrt{a^2-x^2}\,dx = -\tfrac{1}{5}(x^2+\tfrac{2}{3}a^2)\sqrt{(a^2-x^2)^3}$$

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \ln \left(\sqrt{a + bx + cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}} \right) \quad \text{or} \quad \frac{-1}{\sqrt{-c}} \sin^{-1} \left(\frac{b + 2cx}{\sqrt{b^2 - 4ac}} \right)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln (x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

$$\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$$

$$\int x\sqrt{x^2 \pm a^2}\, dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}$$

$$\int x^2\sqrt{x^2 \pm a^2}\, dx = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} x\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln (x + \sqrt{x^2 \pm a^2})$$

$$\int \sin x\, dx = -\cos x$$

$$\int \cos x\, dx = \sin x$$

$$\int \sec x\, dx = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x}$$

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \sin x \cos x\, dx = \frac{\sin^2 x}{2}$$

$$\int \sinh x\, dx = \cosh x$$

$$\int \cosh x\, dx = \sinh x$$

$$\int \tanh x\, dx = \ln \cosh x$$

$$\int \ln x\, dx = x \ln x - x$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int e^{ax}\sin px\,dx = \frac{e^{ax}(a\sin px - p\cos px)}{a^2 + p^2}$$

$$\int e^{ax}\cos px\,dx = \frac{e^{ax}(a\cos px + p\sin px)}{a^2 + p^2}$$

$$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{4 + a^2}\left(a\sin^2 x - \sin 2x + \frac{2}{a}\right)$$

$$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{4 + a^2}\left(a\cos^2 x + \sin 2x + \frac{2}{a}\right)$$

$$\int e^{ax}\sin x\cos x\,dx = \frac{e^{ax}}{4 + a^2}\left(\frac{a}{2}\sin 2x - \cos 2x\right)$$

$$\int \sin^3 x\,dx = -\frac{\cos x}{3}(2 + \sin^2 x)$$

$$\int \cos^3 x\,dx = \frac{\sin x}{3}(2 + \cos^2 x)$$

$$\int \cos^5 x\,dx = \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x$$

$$\int x\sin x\,dx = \sin x - x\cos x$$

$$\int x\cos x\,dx = \cos x + x\sin x$$

$$\int x^2\sin x\,dx = 2x\sin x - (x^2 - 2)\cos x$$

$$\int x^2\cos x\,dx = 2x\cos x + (x^2 - 2)\sin x$$

Radius of curvature

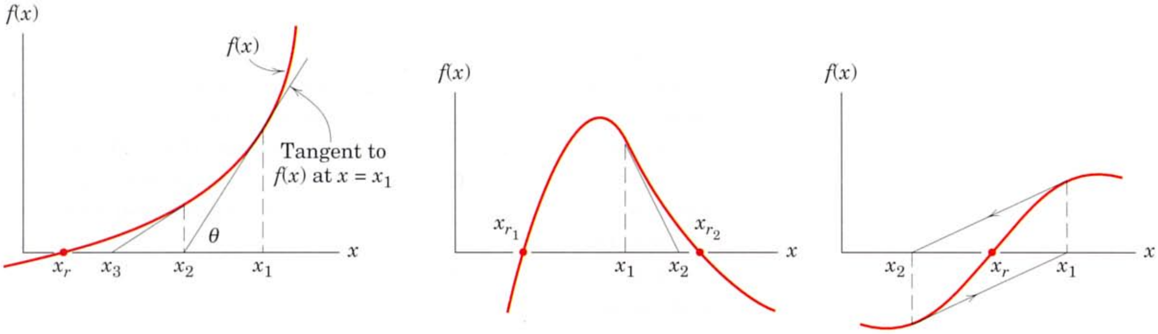
$$\rho_{xy} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

$$\rho_{r\theta} = \frac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$$

C/11 Newton's Method for Solving Intractable Equations

Frequently, the application of the fundamental principles of mechanics leads to an algebraic or transcendental equation which is not solvable (or easily solvable) in closed form. In such cases, an iterative technique, such as Newton's method, can be a powerful tool for obtaining a good estimate to the root or roots of the equation.

Let us place the equation to be solved in the form $f(x) = 0$. Part *a* of the accompanying figure depicts an arbitrary function $f(x)$ for values of x in the vicinity of the desired root x_r. Note that x_r is merely the value



(*a*) (*b*) (*c*)

of x at which the function crosses the x-axis. Suppose that we have available (perhaps via a hand-drawn plot) a rough estimate x_1 of this root. Provided that x_1 does not closely correspond to a maximum or minimum value of the function $f(x)$, we may obtain a better estimate of the root x_r by extending the tangent to $f(x)$ at x_1 so that it intersects the x-axis at x_2. From the geometry of the figure, we may write

$$\tan \theta = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

where $f'(x_1)$ denotes the derivative of $f(x)$ with respect to x evaluated at $x = x_1$. Solving the above equation for x_2 results in

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The term $-f(x_1)/f'(x_1)$ is the correction to the initial root estimate x_1. Once x_2 is calculated, we may repeat the process to obtain x_3, and so forth.

Thus, we generalize the above equation to

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

where

$$\begin{aligned} x_{k+1} &= \text{the } (k + 1)\text{th estimate of the desired root } x_r \\ x_k &= \text{the } k\text{th estimate of the desired root } x_r \\ f(x_k) &= \text{the function } f(x) \text{ evaluated at } x = x_k \\ f'(x_k) &= \text{the function derivative evaluated at } x = x_k \end{aligned}$$

This equation is repeatedly applied until $f(x_{k+1})$ is sufficiently close to zero and $x_{k+1} \cong x_k$. The student should verify that the equation is valid for all possible sign combinations of x_k, $f(x_k)$, and $f'(x_k)$.

Several cautionary notes are in order:

1. Clearly, $f'(x_k)$ must not be zero or close to zero. This would mean, as restricted above, that x_k exactly or approximately corresponds to a minimum or maximum of $f(x)$. If the slope $f'(x_k)$ is zero, then the tangent to the curve never intersects the x-axis. If the slope $f'(x_k)$ is small, then the correction to x_k may be so large that x_{k+1} is a worse root estimate than x_k. For this reason, experienced engineers usually limit the size of the correction term; that is, if the absolute value of $f(x_k)/f'(x_k)$ is larger than a preselected maximum value, that maximum value is used.
2. If there are several roots of the equation $f(x) = 0$, we must be in the vicinity of the desired root x_r in order that the algorithm actually converges to that root. Part *b* of the figure depicts the condition when the initial estimate x_1 will result in convergence to x_{r_2} rather than x_{r_1}.
3. Oscillation from one side of the root to the other can occur if, for example, the function is antisymmetric about a root which is an inflection point. The use of one-half of the correction will usually prevent this behavior, which is depicted in part *c* of the accompanying figure.

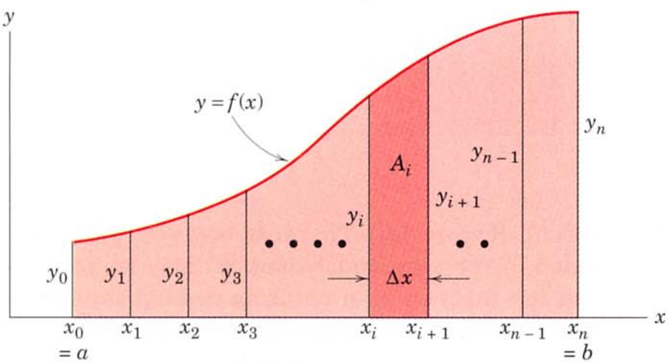
Example: Beginning with an initial estimate of $x_1 = 5$, estimate the single root of the equation $e^x - 10 \cos x - 100 = 0$.

The table below summarizes the application of Newton's method to the given equation. The iterative process was terminated when the absolute value of the correction $-f(x_k)/f'(x_k)$ became less than 10^{-6}.

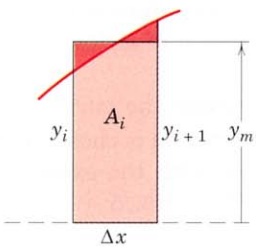
k	x_k	$f(x_k)$	$f'(x_k)$	$x_{k+1} - x_k = -\dfrac{f(x_k)}{f'(x_k)}$
1	5.000 000	45.576 537	138.823 916	−0.328 305
2	4.671 695	7.285 610	96.887 065	−0.075 197
3	4.596 498	0.292 886	89.203 650	−0.003 283
4	4.593 215	0.000 527	88.882 536	−0.000 006
5	4.593 209	$-2(10^{-8})$	88.881 956	$2.25(10^{-10})$

C/12 Selected Techniques for Numerical Integration

1. Area determination. Consider the problem of determining the shaded area under the curve $y = f(x)$ from $x = a$ to $x = b$, as depicted in part a of the figure, and suppose that analytical integration is not feasible. The function may be known in tabular form from experimental measurements or it may be known in analytical form. The function is taken to be continuous within the interval $a < x < b$. We may divide the area into n vertical strips, each of width $\Delta x = (b - a)/n$, and then add the areas of all strips to obtain $A = \int y\,dx$. A representative strip of area A_i is shown with darker shading in the figure. Three useful numerical approximations are cited. In each case the greater the number of strips, the more accurate becomes the approximation geometrically. As a general rule, one can begin with a relatively small number of strips and increase the number until the resulting changes in the area approximation no longer improve the desired accuracy.



(a)



Rectangular

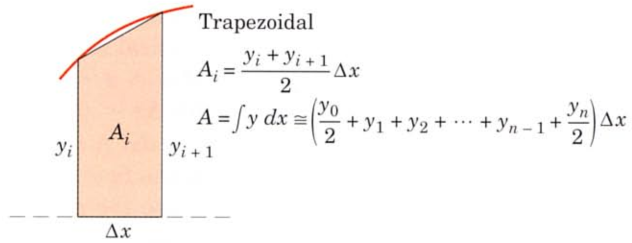
$A_i = y_m \Delta x$

$A = \int y\,dx \cong \Sigma y_m \Delta x$

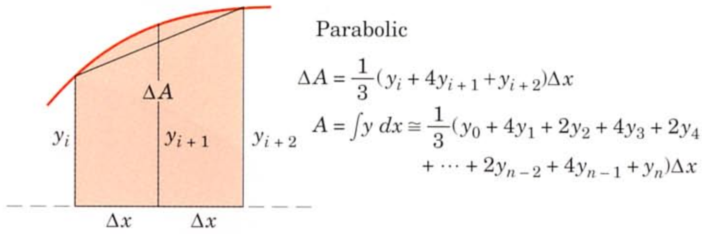
(b)

I. *Rectangular* [Figure (b)] The areas of the strips are taken to be rectangles, as shown by the representative strip whose height y_m is chosen visually so that the small cross-hatched areas are as nearly equal as possible. Thus, we form the sum Σy_m of the effective heights and multiply by Δx. For a function known in analytical form, a value for y_m equal to that of the function at the midpoint $x_i + \Delta x/2$ may be calculated and used in the summation.

II. *Trapezoidal* [Figure (c)] The areas of the strips are taken to be trapezoids, as shown by the representative strip. The area A_i is the average height $(y_i + y_{i+1})/2$ times Δx. Adding the areas gives the area approximation as tabulated. For the example with the curvature shown, clearly the approximation will be on the low side. For the reverse curvature, the approximation will be on the high side.



(*c*)



(*d*)

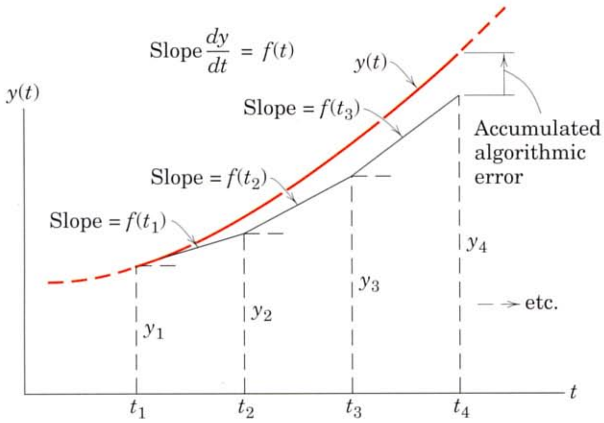
III. *Parabolic* [Figure (*d*)] The area between the chord and the curve (neglected in the trapezoidal solution) may be accounted for by approximating the function by a parabola passing through the points defined by three successive values of y. This area may be calculated from the geometry of the parabola and added to the trapezoidal area of the pair of strips to give the area ΔA of the pair as cited. Adding all of the ΔA's produces the tabulation shown, which is known as Simpson's rule. To use Simpson's rule, the number n of strips must be even.

Example: Determine the area under the curve $y = x\sqrt{1 + x^2}$ from $x = 0$ to $x = 2$. (An integrable function is chosen here so that the three approximations can be compared with the exact value, which is $A = \int_0^2 x\sqrt{1 + x^2}\,dx = \frac{1}{3}(1 + x^2)^{3/2}\Big|_0^2 = \frac{1}{3}(5\sqrt{5} - 1) = 3.393\,447$.)

NUMBER OF SUBINTERVALS	AREA APPROXIMATIONS		
	RECTANGULAR	TRAPEZOIDAL	PARABOLIC
4	3.361 704	3.456 731	3.392 214
10	3.388 399	3.403 536	3.393 420
50	3.393 245	3.393 850	3.393 447
100	3.393 396	3.393 547	3.393 447
1000	3.393 446	3.393 448	3.393 447
2500	3.393 447	3.393 447	3.393 447

Note that the worst approximation error is less than 2 percent, even with only four strips.

2. Integration of first-order ordinary differential equations. The application of the fundamental principles of mechanics frequently results in differential relationships. Let us consider the first-order form $dy/dt = f(t)$, where the function $f(t)$ may not be readily integrable or may be known only in tabular form. We may numerically integrate by means of a simple slope-projection technique, known as Euler integration, which is illustrated in the figure.



Beginning at t_1, at which the value y_1 is known, we project the slope over a horizontal subinterval or step $(t_2 - t_1)$ and see that $y_2 = y_1 + f(t_1)(t_2 - t_1)$. At t_2, the process may be repeated beginning at y_2, and so forth until the desired value of t is reached. Hence, the general expression is

$$y_{k+1} = y_k + f(t_k)(t_{k+1} - t_k)$$

If y versus t were linear, i.e., if $f(t)$ were constant, the method would be exact, and there would be no need for a numerical approach in that case. Changes in the slope over the subinterval introduce error. For the case shown in the figure, the estimate y_2 is clearly less than the true value of the function $y(t)$ at t_2. More accurate integration techniques (such as Runge-Kutta methods) take into account changes in the slope over the subinterval and thus provide better results.

As with the area-determination techniques, experience is helpful in the selection of a subinterval or step size when dealing with analytical functions. As a rough rule, one begins with a relatively large step size and then steadily decreases the step size until the corresponding changes in the integrated result are much smaller than the desired accuracy. A step size which is too small, however, can result in increased error due to a very large number of computer operations. This type of error is generally known as "round-off error," while the error which results from a large step size is known as algorithm error.

Example: For the differential equation $dy/dt = 5t$ with the initial condition $y = 2$ when $t = 0$, determine the value of y for $t = 4$.

Application of the Euler integration technique yields the following results:

NUMBER OF SUBINTERVALS	STEP SIZE	y at $t = 4$	PERCENT ERROR
10	0.4	38	9.5
100	0.04	41.6	0.95
500	0.008	41.92	0.19
1000	0.004	41.96	0.10

This simple example may be integrated analytically. The result is $y = 42$ (exactly).