CH5: Permutation Groups - Problem Session

Question 2: Consider the permutations

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$$

Express α, β, and $\alpha\beta$ as products of disjoint cycles and as products of 2-cycles.

Solution: Finding α in cycle notation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$$

- $1 \to 2 \to 3 \to 4 \to 5 \to 1$ (gives us a 5-cycle)
- $6 \to 7 \to 8 \to 6$ (gives us a 3-cycle)

Disjoint cycle form: $\boxed{\alpha = (12345)(678)}$

As a product of 2-cycles: Using the formula $(a_1 a_2 \ldots a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_2)$:

- $(12345) = (15)(14)(13)(12)$
- $(678) = (68)(67)$

$$\boxed{\alpha = (15)(14)(13)(12)(68)(67)}$$

Finding β in cycle notation:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$$

- $1 \to 1$ (fixed point)
- $2 \to 3 \to 8 \to 4 \to 7 \to 2$ (gives us a 5-cycle)
- $5 \to 6 \to 5$ (gives us a 2-cycle)

Disjoint cycle form: $\boxed{\beta = (23847)(56)}$

As a product of 2-cycles:

- $(23847) = (27)(24)(28)(23)$
- $(56) = (56)$

$$\boxed{\beta = (27)(24)(28)(23)(56)}$$

Finding αβ in cycle notation: **Compute the composition** $(\alpha\beta)(x) = \alpha(\beta(x))$ Apply β first, then α: $\alpha\beta = (12345)(678)(2\ 3\ 8\ 4\ 7)(5\ 6)$

Trace the cycle: $1 \to 2 \to 4 \to 8 \to 5 \to 7 \to 3 \to 6 \to 1$

Disjoint cycle form: $\boxed{\alpha\beta = (12485736)}$

As a product of 2-cycles: $(12485736) = (16)(13)(17)(15)(18)(14)(12)$

$$\boxed{\alpha\beta = (16)(13)(17)(15)(18)(14)(12)}$$

Question3: Write each of the following permutations as a product of disjoint cycles.

a. $(1235)(413)$
b. $(13256)(23)(46512)$
c. $(12)(13)(23)(142)$

Solution: Part a: $(1235)(413)$ **Answer:** $(1235)(413) = (15)(234)$

Part b: $(13256)(23)(46512)$ **Answer:** $(13256)(23)(46512) = (124)(35)$

Part c: $(12)(13)(23)(142)$ **Answer:** $(12)(13)(23)(142) = (1423)$

Question4: Find the order of each of the following permutations.

a. (14)
b. (147)
c. (14762)
d. $(a_1 a_2 \ldots a_k)$

Solution: Part a: (14). This is a 2-cycle (transposition)

- $(14)^2 = \varepsilon$ (identity). **Order = 2**

Part b: (147). This is a 3-cycle

- $(147)^2 = (174)$, $(147)^3 = \varepsilon$. **Order = 3**

Part c: (14762) This is a 5-cycle. **Order = 5**

Part d: $(a_1a_2 \ldots a_k)$. This is a k-cycle. After k applications, each element returns to its original position. **Order =** k

Question5: What is the order of each of the following permutations?

a. $(124)(357)$
b. $(124)(3567)$
c. $(124)(35)$
d. $(124)(357869)$
e. $(1235)(24567)$
f. $(345)(245)$

Solution: Theorem: For disjoint cycles, order = $\text{lcm}(\text{lengths of individual cycles})$

Part a: $(124)(357)$. Lengths: 3 and 3. **Order =** $\text{lcm}(3, 3) = 3$

Part b: $(124)(3567)$. Lengths: 3 and 4. **Order =** $\text{lcm}(3, 4) = 12$

Part c: $(124)(35)$. Lengths: 3 and 2. **Order =** $\text{lcm}(3, 2) = 6$

Part d: $(124)(357869)$. Lengths: 3 and 6. **Order =** $\text{lcm}(3, 6) = 6$

Part e: $(1235)(24567)$. These cycles are NOT disjoint (share element 2). Must compute $(1235)(24567)$ in disjoint form first. Disjoint form: $(1\ 2\ 3\ 6\ 5\ 7\ 4)$. **Order = 7**

Part f: $(345)(245)$. NOT disjoint (share 4 and 5). $(345)(245) = (2453)$. **Order = 4.**

Question7: What is the order of the product of a pair of disjoint cycles of lengths 4 and 6? What about the product of three disjoint cycles of lengths 6, 8, and 10?

Solution: Part 1: Disjoint cycles of lengths 4 and 6. Order = $\mathbf{lcm}(4, 6) = \boxed{12}$

Part 2: Three disjoint cycles of lengths 6, 8, and 10. Order = $\mathbf{lcm}(6, 8, 10)$. Prime factorizations: $6 = 2 \times 3$, $8 = 2^3$, $10 = 2 \times 5$. $\mathbf{lcm} = 2^3 \times 3 \times 5 = \boxed{120}$

Key Principle: For disjoint cycles, the order of their product equals the least common multiple of their individual lengths.

Question8: Determine whether the following permutations are even or odd.

a. (135)
b. (1356)
c. (13567)
d. $(12)(134)(152)$
e. $(1243)(3521)$

Solution: Theorem: An n-cycle can be written as a product of $(n-1)$ transpositions.

- Odd-length cycles → even permutations
- Even-length cycles → odd permutations

Part a: (135). Length 3 (odd) → $3-1=2$ transpositions. **Even permutation**

Part b: (1356). Length 4 (even) → $4-1=3$ transpositions. **Odd permutation**

Part c: (13567). Length 5 (odd) → $5-1=4$ transpositions. **Even permutation**

Part d: $(12)(134)(152)$. (12) → 1 transposition (odd), (134) → 2 transpositions (even), (152) → 2 transpositions (even). Total: $1+2+2=5$ transpositions. **Odd permutation**

Part e: $(1243)(3521)$. (1243) → 3 transpositions (odd), (3521) → 3 transpositions (odd), Total: $3+3=6$ transpositions. **Even permutation**

Question9: Write $((14562)(2345)(136)(235))^{10}$ as a product of disjoint cycles.

Solution:

$$((14562)(2345)(136)(235))^{10} = ((153)(46))^{10} = (153)^{10}(46)^{10} = (153).$$

Question10: Write $(13)(1245)(13)$ and $(24)(13456)(24)$ in disjoint cycle form. Give a simple description of how each product cycle compares with middle cycle in each product.

Solution: $(13)(1245)(13) = (3245)$; $(24)(13456)(24) = (13256)$.

In general, for any cycle a we have $(ij)\alpha(ij)$ is the same as α with i replaced by j.

Question11: Let n be a positive integer. If n is odd, is an n-cycle an odd or an even permutation? If n is even, is an n-cycle an odd or an even permutation?

Solution: An n-cycle can be expressed as $(n-1)$ transpositions:

- $(a_1 a_2 a_3 \ldots a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_3)(a_1 a_2)$

Case 1: n is odd. Number of transpositions $= n - 1 =$ even number. **An n-cycle is an EVEN permutation.**

Case 2: n is even. Number of transpositions = $n - 1$ = odd number. **An n-cycle is an ODD permutation.**

Summary: An n-cycle has the opposite parity of n itself.

Question12: If α is even, prove that α^{-1} is even. If α is odd, prove that α^{-1} is odd.

Proof: Suppose α can be written as a product of k transpositions: $\alpha = \tau_1 \tau_2 \cdots \tau_k$, where each τ_i is a transposition (2-cycle).

Key fact: Each transposition is self-inverse: $\tau_i^2 = \varepsilon$, so $\tau_i^{-1} = \tau_i$

Taking the inverse: $\alpha^{-1} = (\tau_1 \tau_2 \cdots \tau_k)^{-1} = \tau_k^{-1} \cdots \tau_2^{-1} \tau_1^{-1} = \tau_k \cdots \tau_2 \tau_1$

This shows α^{-1} is also a product of k **transpositions** (the same ones, in reverse order).

Conclusion: α and α^{-1} have the same parity. $\square$

Question15: Let α and β belong to S_n. Prove that $\alpha\beta$ is even if and only if α and β are both even or both odd.

Proof: Let α be a product of k transpositions and β be a product of m transpositions. Then $\alpha\beta$ is a product of $k+m$ transpositions.

Case 1: α and β are both even. k is even, m is even. $k+m$ is even. $\alpha\beta$ **is even** ✓

Case 2: α and β are both odd. k is odd, m is odd. $k+m$ is even (odd + odd = even). $\alpha\beta$ **is even** ✓

Case 3: α is even and β is odd (or vice versa). One of k, m is even, the other is odd. $k+m$ is odd (even + odd = odd). $\alpha\beta$ **is odd** ✗

Conclusion: $\alpha\beta$ is even $\Longleftrightarrow$ α and β have the same parity (both even or both odd) □

Analogy: This is like multiplication of ± 1:

- $(+1)(+1) = +1$ (even × even = even)
- $(-1)(-1) = +1$ (odd × odd = even)
- $(+1)(-1) = -1$ (even × odd = odd)

Question17: If n is any integer and α is odd, complete the following statement: α^n is odd if and only if _____.

Solution: Suppose α is a product of k transpositions, where k is odd. Then α^n is a product of nk transpositions. **For α^n to be odd:**

- nk must be odd.
- Since k is odd, we need nk to be odd.
- This happens if and only if n is odd.

Question19: Let α and β belong to S_n. Prove that $\alpha^{-5}\beta\alpha^3$ is odd if and only if β is odd.

Proof:

Step 1: Analyze the parity of $\alpha^{-5}\beta\alpha^3$. Using properties of parity: α^{-5} has the same parity as α^5. α is odd ***iff*** α^5 is odd ***iff*** α^3 is odd

Step 2: Apply the product rule. $\alpha^{-5}\beta\alpha^3$ has parity determined by the number of odd factors:

Case 1: α is even. α^{-5} is even, α^3 is even. Parity of $\alpha^{-5}\beta\alpha^3$ = parity of (even)(β)(even) = parity of β. $\alpha^{-5}\beta\alpha^3$ is odd $\iff$ β is odd ✓

Case 2: α is odd. α^{-5} is odd, α^3 is odd. Parity of $\alpha^{-5}\beta\alpha^3$ = parity of (odd)(β)(odd). $\alpha^{-5}\beta\alpha^3$ is odd $\iff$ β is odd ✓

Key insight: Conjugation preserves parity!

Question24: Find an element in A_8 of order 15. Find an element in A_{12} of order 30.

Solution:

- **Strategy:** Order of disjoint cycles = lcm of cycle lengths
- Need even permutation (in A_n)
- Odd-length cycles are even permutations

Part 1: Element of order 15 in A_8. $\mathbf{15 = 3 \times 5 = lcm(3, 5)}$. Use a 3-cycle and a 5-cycle (disjoint), Both are even permutations (odd lengths). **Answer:** $\alpha = (123)(45678)$.

Part 2: Element of order 30 in A_{12}. $30 = 2 \times 3 \times 5 = \mathrm{lcm}(2,3,5) = \mathrm{lcm}(5,6)$

- Need product of disjoint cycles with lcm = 30
- Must be even permutation

Issue: Single 2-cycle is odd; need multiple even factors. **Solution using lcm(5, 3, 2, 2) = 30:** $\beta = (12345)(678)(9\ 10)(11\ 12)$.

Question25: Let $\beta = (1\ 3\ 5\ 7\ 9\ 8\ 6)(2\ 4\ 10)$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

Solution: $|\beta| = \mathrm{lcm}(7, 3) = 21$. B Thm, $\beta^n = \beta^{-5}$ ***iff*** $n \equiv -5 \pmod{21}$. So $n \equiv 16 \pmod{21}$. So $n = 16$ is the smallest such n.

Question26: What cycle is $(a_1a_2\dots a_n)^{-1}$?

Solution: Determine where each element maps under the inverse.

- **Forward cycle:** $(a_1a_2\dots a_n)$.
- **Inverse must reverse these:**

$$(a_1a_2\dots a_n)^{-1} = \boxed{(a_na_{n-1}\dots a_2a_1)} = (a_1a_na_{n-1}\dots a_2)$$

Alternative notation: Reverse the order of elements in the cycle.

Examples: $(12345)^{-1} = (54321) = (15432)$. $(abc)^{-1} = (cba) = (acb)$

Question27: Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of the members are even.

Proof: Let H be a subgroup of S_n. If all of H are even permutations. Then we're done. Suppose H contains at least one odd permutation, say α.

Define: $E = \{\beta \in H \mid \beta \text{ is even}\}$ (the even permutations in H). **Claim:** $|E| = |H|/2$.

Proof of claim: Define $\phi : E \to H \setminus E$ by $\phi(\beta) = \alpha\beta$

ϕ **is well-defined:** If $\beta \in E$ (even), then $\alpha\beta$ is odd (odd × even = odd). So $\alpha\beta \in H \setminus E$.

ϕ **is injective:** If $\alpha\beta_1 = \alpha\beta_2$, multiply left by α^{-1}: $\beta_1 = \beta_2$

ϕ **is surjective:** Let $\gamma \in H \setminus E$ (γ is odd). Then $\alpha^{-1}\gamma$ is even (odd × odd = even). So $\alpha^{-1}\gamma \in E$ and $\phi(\alpha^{-1}\gamma) = \alpha(\alpha^{-1}\gamma) = \gamma$.

Conclusion: ϕ is a bijection, so $|E| = |H \setminus E|$. Therefore $|E| = |H|/2$ □

Question29: Give two reasons why the set of odd permutations in S_n is not a subgroup.

Solution: Let $O = \{\alpha \in S_n \mid \alpha \text{ is odd}\}$ be the set of odd permutations.

Reason 1: Not closed under composition: Let $\alpha, \beta \in O$ (both odd). Then $\alpha\beta$ has parity: odd × odd = even. So $\alpha\beta \notin O$.

Reason 2: Does not contain the identity: The identity permutation ε is even. $\varepsilon \notin O$.

Question34: Let $H = \{x^2 \mid x \in A_4\}$. Prove that H is not a subgroup of A_4.

Solution: A_4 is given by

$\{\varepsilon, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$

By direct computations of $H = \{x^2 \mid x \in A_4\}$:

- $\varepsilon^2 = \varepsilon$, $(123)^2 = (132)$, $(132)^2 = (123)$, $(124)^2 = (142)$, etc.
- $[(12)(34)]^2 = [(13)(24)]^2 = [(14)(23)]^2 = \varepsilon$

So $H = \{\varepsilon, (123), (132), (124), (142), (134), (143), (234), (243)\}$

Note that $(123)(124) = (13)(24) \notin H$. **Therefore H is not closed, so it is not a subgroup.** $\square$

Question42: If α and β are distinct 2-cycles, what are the possibilities for $|\alpha\beta|$?

Solution: If α and β are disjoint, then $|\alpha\beta| = \mathrm{lcm}(2,2) = 2$. If α and β have exactly one symbol in common we can write $\alpha = (ab)$ and $\beta = (ac)$. Then $\alpha\beta = (ab)(ac) = (acb)$ and $|\alpha\beta| = 3$.

Question51: Prove that S_n is non-Abelian for all $n \geq 3$. Prove that A_n is non-Abelian for all $n \geq 4$.

Solution:

S_n is non-Abelian for $n \geq 3$:

- Note $(12), (23) \in S_n$. $(12)(23) = (123)$, but $(23)(12) = (132)$.

A_n is non-Abelian for $n \geq 4$:

- Note $(123), (124) \in A_n$. $(123)(124) = (13)(24)$, but $(124)(123) = (14)(23)$.

Question55: In S_4, find a cyclic subgroup of order 4 and a noncyclic subgroup of order 4.

Solution: Part 1: For a cyclic subgroup of order 4, need an element of order 4 in S_4 4-cycles have order 4. Let $\alpha = (1234)$ and our subgroup is given by $\langle(1234)\rangle$.

Part 2: Noncyclic subgroup of order 4, need a 4-element subgroup without an element of order 4. Consider $H = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$. **Check it's a subgroup:**

- Contains ε ✓
- Closure:
  - $(12)(34) \cdot (13)(24) = (14)(23)$ ✓
  - $(12)(34) \cdot (14)(23) = (13)(24)$ ✓
  - $(13)(24) \cdot (14)(23) = (12)(34)$ ✓
  - Each element squared is ε ✓
- Inverses: Each element is its own inverse ✓

Check it's noncyclic: Every non-identity element has order 2. So no element of order 4.