Prove that $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy Sequence.

Solution: Given $\epsilon > 0$, we look for $N \in \mathbb{N}$ s.t:

$|x_n - x_m| < \epsilon \quad \forall \ n, m \geq N$.

First assume $n \geq m$ (without Loss of generality)

$$|x_n - x_m| = \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2} \right) \right|$$

$$= \left| \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2} \right|$$

$$= \sum_{k=m+1}^{n} \frac{1}{k^2} \leq \sum_{k=m+1}^{n} \frac{1}{k(k-1)}$$

$$= \sum_{k=m+1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \frac{1}{n} - \frac{1}{m}$$

$$\leq \frac{1}{n} + \frac{1}{m}$$

By AP, for $\epsilon/2 > 0 \ \exists \ N \in \mathbb{N}$ s.t: $\frac{\epsilon}{2} > \frac{1}{N}$

Now for $n, m \geq N \Rightarrow |x_n - x_m| \leq \frac{1}{n} + \frac{1}{m}$

$$\leq \frac{1}{N} + \frac{1}{N}$$

$$= \frac{2}{N} < \epsilon.$$