Chapter 3: Capacitance and Dielectrics

3.1 Capacitors

If we have two isolated conductors close to each other with +Q charge on one and −Q on the other, a potential difference ΔV will exist between them.

The quantity of charge Q is linearly proportional to the potential difference between the conductors

The capacitance C of these isolated conductors is defined to be:

$$C = \frac{Q}{\Delta V}$$

By definition, the capacitance is always a positive quantity.

The potential difference between the two conductors can be given be:

$$V = V_{+} - V_{-} = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}.$$

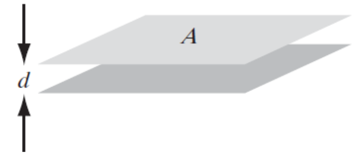
The unit of capacitance is the farad, or coulomb per volt

$$1F = 1\frac{C}{V}$$

<u>Capacitance is a purely geometrical quantity. It is determined by the sizes, shapes, and separation of the two conductors.</u>

Example 3.1

Find the capacitance of a parallel-plate capacitor consisting of two metal surfaces of area A held a distance d apart.



from chapter 2 E between the two plates is

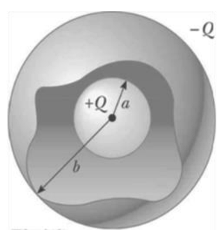
$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{\epsilon_0 A}$$

$$V = -\int_d^0 E\, dl \Rightarrow V = \frac{qd}{\epsilon_0 A}$$

$$\Rightarrow C = \frac{q}{V} = \epsilon_0 \frac{A}{d}$$

Example 3.2

Find the capacitance of two concentric spherical metal shells, with radii *a* and *b*.



from Gauss's law the electric field between the spheres is:

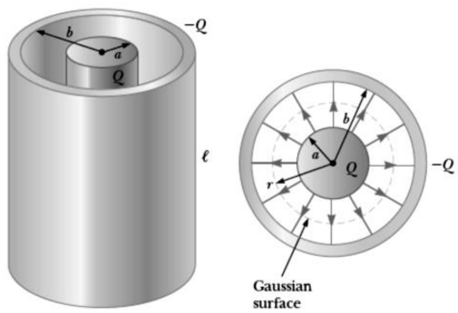
$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

$$V = -\int_b^a \vec{E} \cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0} \int_b^a \frac{Q}{r^2} \hat{r} = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$\Rightarrow \quad C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{(b-a)}$$

Example 3.3

A solid cylindrical conductor of radius a and charge Q is coaxial with a cylindrical shell of negligible thickness, radius b > a, and charge -Q. Find the capacitance of this cylindrical capacitor if its length is l



$Q = \lambda \ell$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V_b - V_a = -\int_a^b \frac{\lambda}{2\pi\epsilon_0 r} dr = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$V_a > V_b$

$$C = \frac{Q}{V} = \frac{\lambda \ell}{\left(\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)\right)} \Rightarrow C = \frac{2\pi\epsilon_0 \ell}{\ln\left(\frac{b}{a}\right)}$$

the capacitance per unit length $= \dfrac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$

3.2 Combinations of Capacitors

3.2.1 Parallel Combination

For two capacitors connected in parallel:

$$\Delta V = \Delta V_1 = \Delta V_2$$

$$Q = Q_1 + Q_2$$

$$C\,\Delta V = C_1\,\Delta V + C_2\,\Delta V$$

$$C = C_1 + C_2$$

In general,

$$C = C_1 + C_2 + C_3 + \ldots\ldots..$$

3.2.2 Series Combination

For two capacitors connected in series:

$$Q = Q_1 = Q_2$$

$$\Delta V = \Delta V_1 + \Delta V_2$$

$$Q / C = Q / C_1 + Q / C_2$$

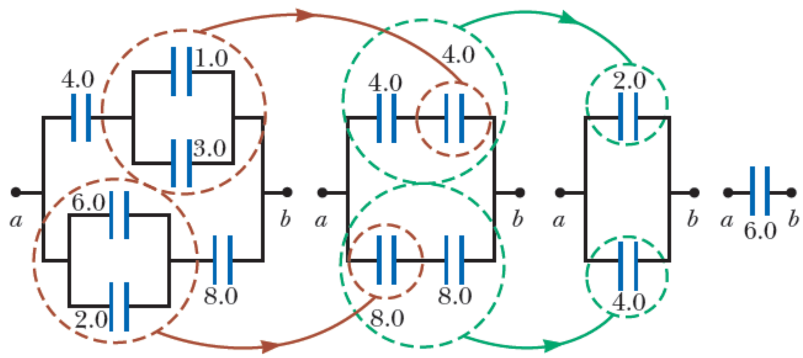
$$1 / C = 1 / C_1 + 1 / C_2$$

In general,

$$1 / C = 1 / C_1 + 1 / C_2 + 1 / C_3 + \ldots$$

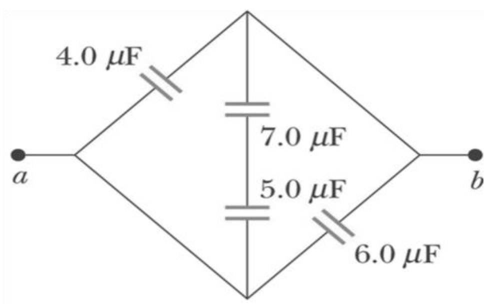
Example 3.4

Find the equivalent capacitance between points *a* and *b* for the combination of capacitors shown in the following figure. All capacitances are in microfarads.



Example 3.5

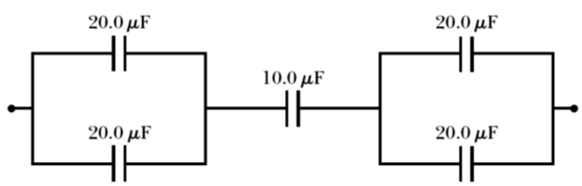
Find the equivalent capacitance between a and b



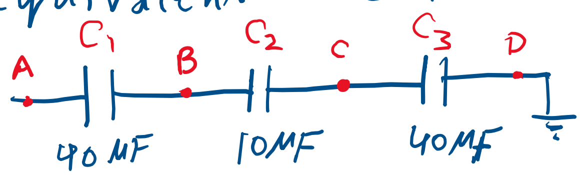
<u>C = 12.9 μF</u>

Example 3.6

Each capacitor in the combination shown in the figure below has a breakdown voltage of 15V. What is the breakdown voltage of the combination?



The equivalent circuit



we have the same charge Q on each capacitor

$$\Rightarrow Q = (40\,\mu F)\,\Delta V_{AB} = (10\,\mu F)\,\Delta V_{BC} = (40\,\mu F)\,\Delta V_{CD}$$

$$4\,\Delta V_{AB} = \Delta V_{BC} = 4\,\Delta V_{CD}$$

C_2 will break down first at 15 V

at this $\Delta V_{AB} = \Delta V_{CD} = \frac{15}{4} = 3.75\ V$

$$\Rightarrow \Delta V_{AD} = 3.75 + 15 + 3.75 = 22.5\ V$$

Example 3.7

Find the force with which the plates of a parallel-plate capacitor attract each other. Also determine the pressure on the surface of the plate due to the field.

The electric field intensity on each plate of the capacitor is: $E = \frac{\sigma}{2\epsilon_0}$

$$F = QE = Q\,\frac{Q}{2A\epsilon_0} = \frac{Q^2}{2A\epsilon_0}$$

$$P = F/A = \frac{Q^2}{2A^2\epsilon_0} = \frac{\sigma^2}{2\epsilon_0}$$

Example 3.8

A parallel-plate capacitor has plate area 200 cm^2 and plate separation 3 mm. The charge density is 1 $\mu C/m^2$ with air as dielectric. Find:
(a) The capacitance of the capacitor
(b) The voltage between the plates
(c) The force with which the plates attract each other

a) $C = \epsilon_0 \frac{A}{d} = \frac{8.85 \times 10^{-12} \times 200 \times 10^{-4}}{3 \times 10^{-3}}$

$= 59 \, pF$

b) $E = \frac{V}{d} = \frac{\sigma}{\epsilon_0} \Rightarrow V = \frac{3 \times 10^{-3} \times 1 \times 10^{-6}}{8.85 \times 10^{-12}} \simeq 339 \, V$

c) $F = \frac{Q^2}{2A\epsilon_0} = \frac{\sigma^2 A}{2\epsilon_0} = 1.13 \, mN$

3.3 Energy stored in a charged capacitor

The work necessary to transfer an increment of charge dq from the plate carrying charge - Q to the plate carrying charge Q is:

$$dW = \Delta V \, dq = (q/C)dq$$

$$U = W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

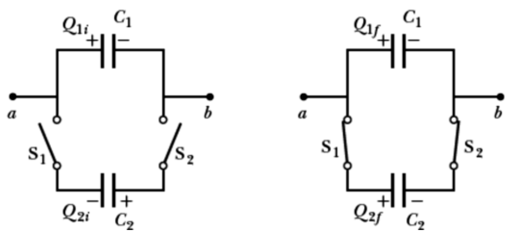
$$U = \frac{Q^2}{2C} = \frac{1}{2} Q\Delta V = \frac{1}{2} C(\Delta V)^2$$

Example 3.9

Two capacitors C_1 and C_2 (where $C_1 > C_2$) are charged to the same initial potential difference ΔV_i. The charged capacitors are removed from the battery, and their plates are connected with opposite polarity as in the figure below.

a) Find the final potential difference ΔV_f between *a* and *b* after the switches are closed.

b) Find the total energy stored in the capacitors before and after the switches are closed and determine the ratio of the final energy to the initial energy.



a) The total charge on the left hand plates:

$$Q_i = Q_{1i} + Q_{2i} = C_1 \Delta V_{1i} - C_2 \Delta V_{2i} = (C_1 - C_2) V_i$$

$$Q_f = Q_{1f} + Q_{2f} = C_1 \Delta V_{1f} + C_2 \Delta V_{2f} = (C_1 + C_2) V_f$$

The system is isolated $\Rightarrow Q_i = Q_f$

$$\Rightarrow (C_1 + C_2)\Delta V_f = (C_1 - C_2)\Delta V_i$$

$$\Rightarrow \Delta V_f = \left(\frac{C_1 - C_2}{C_1 + C_2}\right) V_i$$

b) $$U_i = \frac{1}{2} C_1 (\Delta V_i)^2 + \frac{1}{2} C_2 (\Delta V_i)^2$$

$$= \frac{1}{2}(C_1 + C_2)(\Delta V_i)^2$$

$$U_f = \frac{1}{2} C_1 (\Delta V_f)^2 + \frac{1}{2} C_2 (\Delta V_f)^2$$

$$= \frac{1}{2}(C_1 + C_2)(\Delta V_f)^2$$

$$U_f = \frac{1}{2} \frac{(C_1 - C_2)^2}{C_1 + C_2} (\Delta V_i)^2$$

$$\frac{U_f}{U_i} = \left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2$$ less than unity?

3.4 Dielectrics

- Dielectrics are nonconducting materials in which all electrons are bound and they cannot move freely through the material under the influence of an electric field.

- The only possible motion of charges in dielectrics in an electric field is a very small displacement of positive and negative charges in opposite directions.

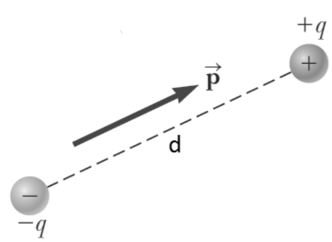
- A dielectric in which this charge displacement has taken place is said to be polarized, and its molecules are said to possess induced dipole moments.

- These dipoles produce their own electric field.

- An applied electric field can also orient molecules that possess permanent dipole moments.

3.5 Electric dipole in an electric field

The electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance d

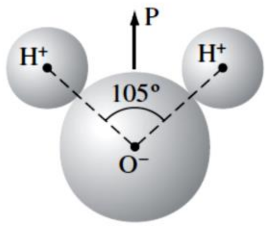


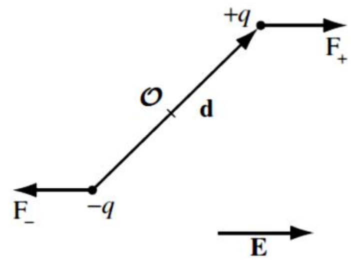
The electric dipole moment of this configuration is defined as the vector **p** directed from the negative charge -q toward the positive charge +q along the line joining the charges and having magnitude:

$$\mathbf{p} = q\,\mathbf{d}$$

Nonpolar molecules have no built-in dipole moments. An applied electric field can induce dipoles in them.

Polar molecules have built-in, permanent dipole, see the following figure of water molecule





The polar molecules are randomly oriented in the absence of an electric field. When an external field is applied, a torque is exerted on the dipoles, causing them to partially align with the field

$$\boldsymbol{\tau} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$
$$= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}$$

$$\Rightarrow \quad \boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$$

3.6 The electric polarization P

The electric polarization p is **the dipole moment per unit volume** at a given point.

$$\mathbf{P} = N\,\mathbf{p}$$

Where **p** is the average electric dipole moment per molecule
and **N** is the number of molecules per unit volume.

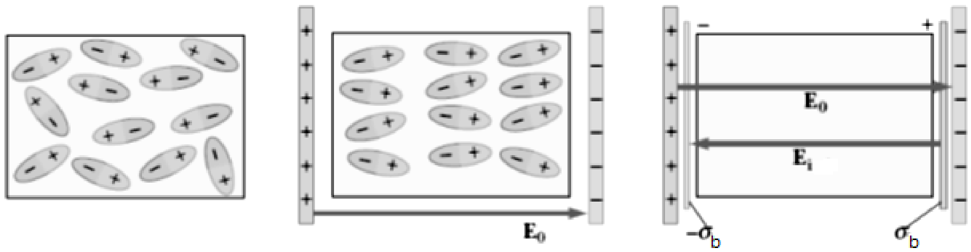
Therefore, the total dipole moment in material with volume V is:

$$p = \int_V P\,d\tau$$

3.7 Free and bound charges

Polarization causes charges to accumulate, either within the dielectric or at its surface. We refer to such charges as bound charges. Other charges are said to be free.

- The figure below shows how the polar molecules are randomly oriented in the absence of an external electric field
- When an external field is applied the molecules are partially oriented that creates an induced electric filed inside the dielectric in the opposite direction of the external electric field
- This figure shows the bound charges Q_b on the surface of the dielectric with surface charge density of σ_b.



To calculate the actual amount of bound charge resulting from the polarization, let us take the piece of dielectric shown in this figure with area A and width d:

The total dipole moment in material:

$$\mathbf{p} = \mathbf{P}(Ad)$$

Therefore, the bound charge is

$$Q_b = PA$$

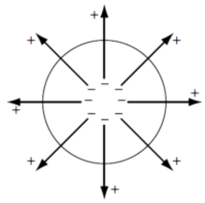
The surface charge density of the bound charge is

$$\sigma_b = Q_b/A = P$$

In general, if the surface area and the polarization vectors are not parallel as shown in the figure below:

$$\sigma_b = \frac{Q_b}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}.$$

If the polarization is nonuniform, we get accumulations of bound charge within the material, as well as on the surface



This figure shows that diverging **P** results in accumulation of negative charges, therefore, the net bound charge in a given volume is equal and <u>opposite</u> to the amount that has been pushed out through the surface ($\sigma_b = P$).

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{a}$$

From the divergence theorem:

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{P}) \, d\tau.$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

3.8 Gauss's Law in the Presence of Dielectrics

Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f$$

From Gauss's low in its differential form:

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

where E is now the total field $\mathbf{E} = \mathbf{E}_o - \mathbf{E}_i$

combine the two divergence terms

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f .$$

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},}$$

D is known as the electric displacement or electric flux density.
In terms of D, Gauss's law reads

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f ,}$$

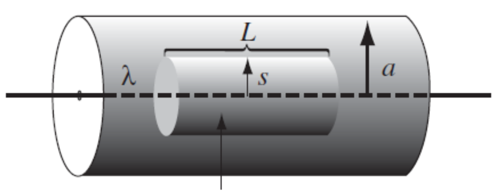
In integral form

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\mathrm{enc}}}$$

where $Q_{f\,enc}$ denotes the total free charge enclosed in the volume.
The flux of the electric displacement D through a closed surface (of a volume that lies completely inside a dielectric) is equal to the free charge enclosed by the surface.
Note that the surface integral of D, are unaffected by the bound charges.

Example 3.10

A long straight wire, carrying uniform line charge λ, is surrounded by cylindrical rubber insulation out to a radius a. Find the electric displacement



Gaussian surface

$$\oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}$$

$$D(2\pi s L) = \lambda L \Rightarrow \vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

$s < a$ we need $\vec{P}$ to determine $\vec{E}$

$s > a$ $\vec{P} = 0 \Rightarrow \vec{E} = \frac{D}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$

3.9 Susceptibility, Permittivity, Dielectric Constant in linear dielectrics

In liner dielectrics the molecular charge separation is directly proportional to, and in the same direction as, E.

The polarization P in liner dielectrics is proportional to the field, provided E is not too strong

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}.$$

where χ_e, known as the electric susceptibility of the material, which can be a measure of how susceptible (or sensitive) a given dielectric is to electric fields. (a factor of ϵ_o has been extracted to make χ_e dimensionless)

In linear media we have:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E},$$

D is also proportional to E:

$$\mathbf{D} = \epsilon \mathbf{E},$$

where,

$$\epsilon \equiv \epsilon_0 (1 + \chi_e).$$

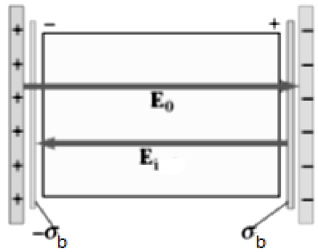
ϵ is called the permittivity of the material

Also the relative permittivity ϵ_r of a dielectric (was formerly called the dielectric constant) is defined as:

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

Example 3.11

Show that the capacitance of a parallel plate capacitor can be increased by inserting dielectric between the plates



$$\vec{E} = \vec{E}_0 - \vec{E}_i$$

Before inserting dielectric $\quad C_0 = \frac{q}{V_0} = \frac{q}{E_0 d}$

After inserting dielectric $\quad C = \frac{q}{V} = \frac{q}{E d}$

$$\boxed{\frac{C}{C_0} = \frac{E_0}{E} = \frac{\epsilon}{\epsilon_0} = \epsilon_r} \Rightarrow C > C_0 \Rightarrow C = \epsilon_r C_0$$

$$E = \frac{\sigma}{\epsilon} \qquad E_0 = \frac{\sigma}{\epsilon_0}$$

Example 3.12

(a) How much charge can be placed on a capacitor with air between the plates before it breaks down if the area of each plate is 5 cm^2? The breakdown electric field (dielectric strength) in air $E_{max} = 3x10^6$ V/m
(b) Find the maximum charge if polystyrene (ϵ_r= 2.56 , $E_{max} = 24x10^6$ V/m) is used between the plates instead of air.

a) $Q_{max} = C \Delta V_{max}$

$\Delta V_{max} = E_{max} d$ $\qquad C = \dfrac{\epsilon_r \epsilon_o A}{d}$

$$\Rightarrow Q_{max} = \frac{\epsilon_r \epsilon_o A}{d}(E_{max} d) = \epsilon_r \epsilon_o A E_{max}$$

$\epsilon_r (air) = 1$

$$\Rightarrow Q_{max} = 8.85 \times 10^{-12} \times 5 \times 10^{-4} \times 3 \times 10^{6}$$
$$= 13.3 \text{ nC}$$

b) $Q_{max} = \epsilon_r \epsilon_o A E_{max}$

$$= 2.56 (8.85 \times 10^{-12})(5 \times 10^{-4})(24 \times 10^{6})$$

$$= 272 \text{ nC}$$

Example 3.14

The space between the plates of a parallel-plate capacitor with a plate separation 1 cm and a surface area 100 cm^2 is filled with a 0.5 cm-thick dielectric slab. If the relative permittivity of the dielectric is 7 and the potential difference in the absence of the dielectric is 100 V find:

a) The capacitance before inserting the dielectric
b) The free charge on each capacitor plate
c) The electric field before inserting the dielectric
d) The induced electric field within the dielectric
e) The potential difference after inserting the dielectric
f) The capacitance after inserting the dielectric
g) The polarization and electric displacement before and after inserting the dielectric

a) $C_o = \dfrac{\epsilon_o A}{d} = \dfrac{8.85\times10^{-12} \times 100\times10^{-4}}{1\times10^{-2}} = 8.85 \text{ pF}$

b) $C_o = \dfrac{q_f}{V_o} \Rightarrow q_f = C_o V_o = 8.85\times10^{-12} \times 100 = 8.85\times10^{-10} C$

c) $E_o = \dfrac{V_o}{d} = \dfrac{100}{1\times10^{-2}} = 1\times10^{4} \text{ V/m}$

d) $E = E_o - E_i \Rightarrow E_i = E_o - E = E_o - \dfrac{E_o}{\epsilon_r}$

$E_i = 1\times10^{4} - 0.143\times10^{4} = 8.57\times10^{3} \text{ V/m}$

e) $V = -\int \vec{E}\cdot d\vec{\ell} = E_o(d-s) + Es$

$= 1\times10^{4} \times (1\times10^{-2} - 0.5\times10^{-2}) + 1.43\times10^{3}\times0.5\times10^{-2}$

$\Rightarrow V = 57 \text{ V}$

f) $C = \dfrac{q_f}{V} = \dfrac{8.85\times10^{-10}}{57} = 15.5\ pF$

we cannot use $\underline{C = \epsilon_r C_o}$ here why?

g) $P = \epsilon_o \chi_e E$

$D = \epsilon E = \epsilon_r \epsilon_o E$

<u>with dielectric:</u>

$P = 8.85\times10^{-12}(7-1)(1.43\times10^{3}) \simeq 7.6\times10^{-8}\ C/m^2$

$$\boxed{P = \sigma_b}$$

$D = \epsilon_r \epsilon_o E = 7(8.85\times10^{-12})(1.43\times10^{3}) = 8.85\times10^{-8}\ C/m^2$

<u>without dielectric</u>

$D_o = \epsilon_r \epsilon_o E_o = 1(8.85\times10^{-12})(1\times10^{4})$

$\Rightarrow D_o = 8.85\times10^{-8}\ C/m^2 = D = \sigma_f$

$P_o = 0$ why?