جامعة
الملك سعود
King Saud University

MATH 111- Final Exam- Semester 2- 1447H

Time: 3 hours

Instructions: Calculators are **not** allowed.

Question 1 (2+2+2 Marks):

1. Find the value of c that satisfies the mean value theorem for definite integrals for the function $f(x) = (x+1)^2$ on the interval $[-1, 2]$.

2. If $F(x) = \int_{\sec x}^{\log_2 x} \frac{1}{\sqrt{1+t}}\, dt$, find $F'(x)$.

3. If $y(x) = \ln\left|\frac{(x^2+1)^{\frac{2}{3}}\cos^7(5x)}{\sqrt[3]{x^4+1}}\right|$, find $\frac{dy}{dx}$.

Question 2: Evaluate the following integrals (2+2+3+3+3+3 Marks)

1. $\int \frac{1}{x + x\ln x}\, dx$
2. $\int \frac{\sec x \tan x}{1+\sec^2 x}\, dx$
3. $\int e^{2x}\sin(x)\, dx$
4. $\int \frac{3x^2+1}{(x^2-1)(x^2+1)}\, dx$
5. $\int \frac{1}{x^2\sqrt{9-x^2}}\, dx$
6. $\int \frac{1}{(x+3)\sqrt{x^2+6x+13}}\, dx.$

Question 3 (2.5+2+3+3+2.5+2+3 Marks):

1. Calculate $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{2x}$.

2. Determine whether the improper integral $\int_0^2 \frac{1}{\sqrt{2-x}}\, dx$ converges or diverges.

3. Sketch the region bounded by the graphs of the curves $y = 2 - x^2$ and $y = x$, then find its area.

4. Sketch the region bounded by the graphs of the curves $y = \frac{x^2}{2}$ and $y = 2$, then find the volume of the solid generated by revolving this region about the $x-$axis.

5. Find the arc length of $y = \ln(\sec x)$, from $x = 0$ to $x = \frac{\pi}{4}$.

6. Convert the polar equation $\frac{3}{r} = \sin\theta - \sec\theta$ into a Cartesian equation.

7. Sketch the region in the first quadrant, inside the curve $r = 1 + \sin\theta$ and outside the curve $r = 1$, then find its area.

MATH 111 - Integral Calculus

Second Semester - 1447 H

Solution of the Final Exam

Dr Tariq A. Alfadhel

Question (1): [$2+2+2=6$ marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x)=(x+1)^2$ on the interval $[-1,2]$.

   Solution: Using the formula $(b-a)\, f(c)=\int_a^b f(x)\, dx$.

   $$(2-(-1))\,(c+1)^2=\int_{-1}^{2}(x+1)^2\, dx=\left[\frac{(x+1)^3}{3}\right]_{-1}^{2}$$

   $$3(c+1)^2=\frac{(2+1)^3}{3}-\frac{(-1+1)^3}{3}=\frac{27}{3}-\frac{0}{3}=9$$

   $$(c+1)^2=3 \implies c+1=\pm\sqrt{3} \implies c=-1\pm\sqrt{3}\ .$$

   Note that $c=-1-\sqrt{3}\notin(-1,2)$ and $c=-1+\sqrt{3}\in(-1,2)$.

   The desired value is $c=-1+\sqrt{3}$.

2. Find $F'(x)$, if $F(x)=\int_{\sec x}^{\log_2 x}\frac{1}{\sqrt{1+t}}\, dt$.

   Solution:

   $$F'(x)=\frac{d}{dx}\int_{\sec x}^{\log_2 x}\frac{1}{\sqrt{1+t}}\, dt$$

   $$=\frac{1}{\sqrt{1+\log_2 x}}\left(\frac{1}{x}\,\frac{1}{\ln 2}\right)-\frac{1}{\sqrt{1+\sec x}}\,(\sec x\tan x)$$

   $$=\frac{1}{x\,\ln 2\,\sqrt{1+\log_2 x}}-\frac{\sec x\tan x}{\sqrt{1+\sec x}}\ .$$

3. Find y' if $y(x)=\ln\left|\frac{(x^2+1)^{\frac{2}{3}}\cos^7(5x)}{\sqrt[3]{x^4+1}}\right|$.

   Solution:

   $$y(x)=\ln\left|(x^2+1)^{\frac{2}{3}}\cos^7(5x)\right|-\ln\left|\sqrt[3]{x^4+1}\right|$$

   $$y(x)=\ln\left|(x^2+1)^{\frac{2}{3}}\right|+\ln\left|\cos^7(5x)\right|-\ln\left|(x^4+1)^{\frac{1}{3}}\right|\ .$$

   $$y(x)=\frac{2}{3}\ln|x^2+1|+7\ln|\cos(5x)|-\frac{1}{3}\ln|x^4+1|$$

$$y'(x)=\frac{2}{3}\left(\frac{2x}{x^2+1}\right)+7\left(\frac{-\sin(5x)\,(5)}{\cos(5x)}\right)-\frac{1}{3}\left(\frac{4x^3}{x^4+1}\right)$$

Question (2): [2 + 2 + 3 + 3 + 3 + 3 = 16 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{1}{x+x\ln|x|}\,dx$.

   Solution:

$$\int \frac{1}{x+x\ln|x|}\,dx=\int \frac{1}{x\,(1+\ln|x|)}\,dx$$
$$=\int \frac{\left(\frac{1}{x}\right)}{1+\ln|x|}\,dx=\ln|1+\ln|x||+c\ .$$

2. $\displaystyle\int \frac{\sec x\tan x}{1+\sec^2 x}\,dx$

   Solution:

$$\int \frac{\sec x\tan x}{1+\sec^2 x}\,dx=\int \frac{\sec x\tan x}{(1)^2+(\sec x)^2}\,dx=\tan^{-1}(\sec x)+c\ .$$

3. $\displaystyle\int e^{2x}\sin x\,dx$.

   Solution: Using integration by parts .

$$u=\sin x \qquad dv=e^{2x}\,dx$$
$$du=\cos x\,dx \qquad v=\frac{1}{2}\,e^{2x}$$

$$\int e^{2x}\sin x\,dx=\sin x\left(\frac{1}{2}\,e^{2x}\right)-\int \frac{1}{2}e^{2x}\,\cos x\,dx$$
$$=\frac{1}{2}\,e^{2x}\sin x-\frac{1}{2}\int e^{2x}\cos x\,dx$$

Using integration by parts again .

$$u=\cos x \qquad dv=e^{2x}\,dx$$
$$du=-\sin x\,dx \qquad v=\frac{1}{2}\,e^{2x}$$

$$\int e^{2x}\sin x\,dx=\frac{1}{2}\,e^{2x}\sin x-\frac{1}{2}\left[\frac{1}{2}\,e^{2x}\cos x-\int \frac{1}{2}e^{2x}\,(-\sin x)\,dx\right]$$
$$=\frac{1}{2}\,e^{2x}\sin x-\frac{1}{4}\,e^{2x}\cos x-\frac{1}{4}\int e^{2x}\,\sin x\,dx$$
$$\int e^{2x}\sin x\,dx+\frac{1}{4}\int e^{2x}\sin x\,dx=\frac{1}{2}\,e^{2x}\sin x-\frac{1}{4}\,e^{2x}\cos x+c$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}\,e^{2x}\sin x - \frac{1}{4}\,e^{2x}\cos x + c$$

$$\int e^{2x}\sin x\,dx = \frac{4}{5}\left[\frac{1}{2}\,e^{2x}\sin x - \frac{1}{4}\,e^{2x}\cos x + c\right].$$

4. $\displaystyle\int \frac{3x^2+1}{(x^2-1)(x^2+1)}\,dx$.

Solution : Using the method of partial fractions.

$$\frac{3x^2+1}{(x^2-1)(x^2+1)} = \frac{3x^2+1}{(x-1)(x+1)(x^2+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{Bx+C}{x^2+1}$$

$$3x^2+1 = A_1(x+1)(x^2+1) + A_2(x-1)(x^2+1) + (Bx+C)(x^2-1)$$

$$3x^2+1 = A_1x^3 + A_1x + A_1x^2 + A_1 + A_2x^3 + A_2x - A_2x^2 - A_2$$
$$+Bx^3 - Bx + Cx^2 - C$$

$$3x^2+1 = (A_1+A_2+B)\,x^3 + (A_1-A_2+C)\,x^2 + (A_1+A_2-B)\,x$$
$$+(A_1-A_2-C)$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{ll} A_1+A_2+B=0 & \longrightarrow\ (1) \\ A_1-A_2+C=3 & \longrightarrow\ (2) \\ A_1+A_2-B=0 & \longrightarrow\ (3) \\ A_1-A_2-C=1 & \longrightarrow\ (4) \end{array}$$

Equation (1) - Equation (3) : $2B=0 \implies B=0$.

Equation (2) - Equation (4) : $2C=2 \implies C=1$.

Equation (1) becomes : $A_1+A_2=0 \longrightarrow (5)$.

Equation (2) becomes : $A_1-A_2=2 \longrightarrow (6)$.

Equation (5) + Equation (6) : $2A_1=2 \implies A_1=1$.

From Equation (5) : $1+A_2=0 \implies A_2=-1$

$$\int \frac{3x^2+1}{(x^2-1)(x^2+1)}\,dx = \int\left(\frac{1}{x-1} + \frac{-1}{x+1} + \frac{1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x-1| - \ln|x+1| + \tan^{-1}x + c\ .$$

5. $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$.

Solution : Using trigonometric substitutions.

Put $x = 3\sin\theta \implies \sin\theta = \frac{x}{3}$.

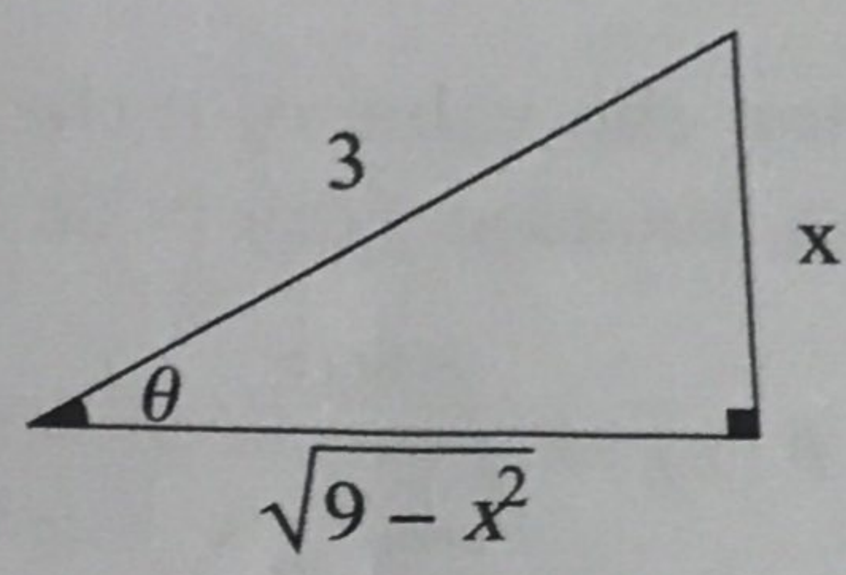
$dx = 3\cos\theta \; d\theta$.

$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$$\int \frac{1}{x^2\sqrt{9 - x^2}} \, dx = \int \frac{3\cos\theta}{(3\sin\theta)^2 \; 3\cos\theta} \, d\theta = \int \frac{1}{9\sin^2\theta} \, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta} d\theta = \frac{1}{9}\int \csc^2\theta \; d\theta = -\frac{1}{9}\cot\theta + c$$

$\sin\theta = \frac{x}{3}$



From the triangle :

$\cot\theta = \frac{\sqrt{9 - x^2}}{x}$

$$\int \frac{1}{x^2\sqrt{9 - x^2}} \, dx = -\frac{1}{9}\frac{\sqrt{9 - x^2}}{x} + c \, .$$

6. $\displaystyle\int \frac{1}{(x+3)\sqrt{x^2 + 6x + 13}} \, dx$.

Solution : By completing the square.

$x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x+3)^2 + (2)^2$

$$\int \frac{1}{(x+3)\sqrt{x^2 + 6x + 13}} \, dx = \int \frac{1}{(x+3)\sqrt{(x+3)^2 + (2)^2}} \, dx$$

$$= -\frac{1}{2}\text{csch}^{-1}\left(\frac{x+3}{2}\right) + c \, .$$

Question (3): [2.5 + 2 + 3 + 3 + 2.5 + 2 + 3 = 18 marks]

1. Calculate $\displaystyle\lim_{x\to\infty}\left(1 + \frac{4}{x}\right)^{2x}$.

Solution:

$$\lim_{x\to\infty}\left(1 + \frac{4}{x}\right)^{2x} \qquad (1^{\infty}) \, .$$

$$\lim_{x\to\infty}\left(1 + \frac{4}{x}\right)^{2x} = \lim_{x\to\infty} e^{\ln\left|1 + \frac{4}{x}\right|^{2x}} = \lim_{x\to\infty} e^{2x\ln\left|1 + \frac{4}{x}\right|}$$

$$\lim_{x\to\infty} 2x\ln\left|1 + \frac{4}{x}\right| \qquad (\infty . 0) \, .$$

$$= \lim_{x\to\infty} \frac{\ln\left|1+\frac{4}{x}\right|}{\left(\frac{1}{2x}\right)} \quad \left(\frac{0}{0}\right).$$

Using L'Hôpital's rule.

$$\lim_{x\to\infty} \frac{\left(\frac{4\left(\frac{-1}{x^2}\right)}{\left(1+\frac{4}{x}\right)}\right)}{\left(\frac{1}{2}\,\frac{-1}{x^2}\right)} = \lim_{x\to\infty} \frac{8}{\left(1+\frac{4}{x}\right)} = \frac{8}{1} = 8 .$$

Therefore, $\lim_{x\to\infty} 2x \ln\left|1+\frac{4}{x}\right| = 8$.

So, $\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{2x} = e^8$.

2. Determine whether the improper integral $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$ converges or diverges.

Solution:

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{t\to 2^-} \int_0^t (2-x)^{-\frac{1}{2}}\,dx = \lim_{t\to 2^-} \left((-1)\int_0^t (2-x)^{-\frac{1}{2}}(-1)\,dx\right)$$

$$= \lim_{t\to 2^-} \left((-1)\left[\frac{(2-x)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^t\right) = \lim_{t\to 2^-} \left((-1)\left[2\sqrt{2-t} - 2\sqrt{2-0}\right]\right)$$

$$= (-1)\left[0 - 2\sqrt{2}\right] = 2\sqrt{2} .$$

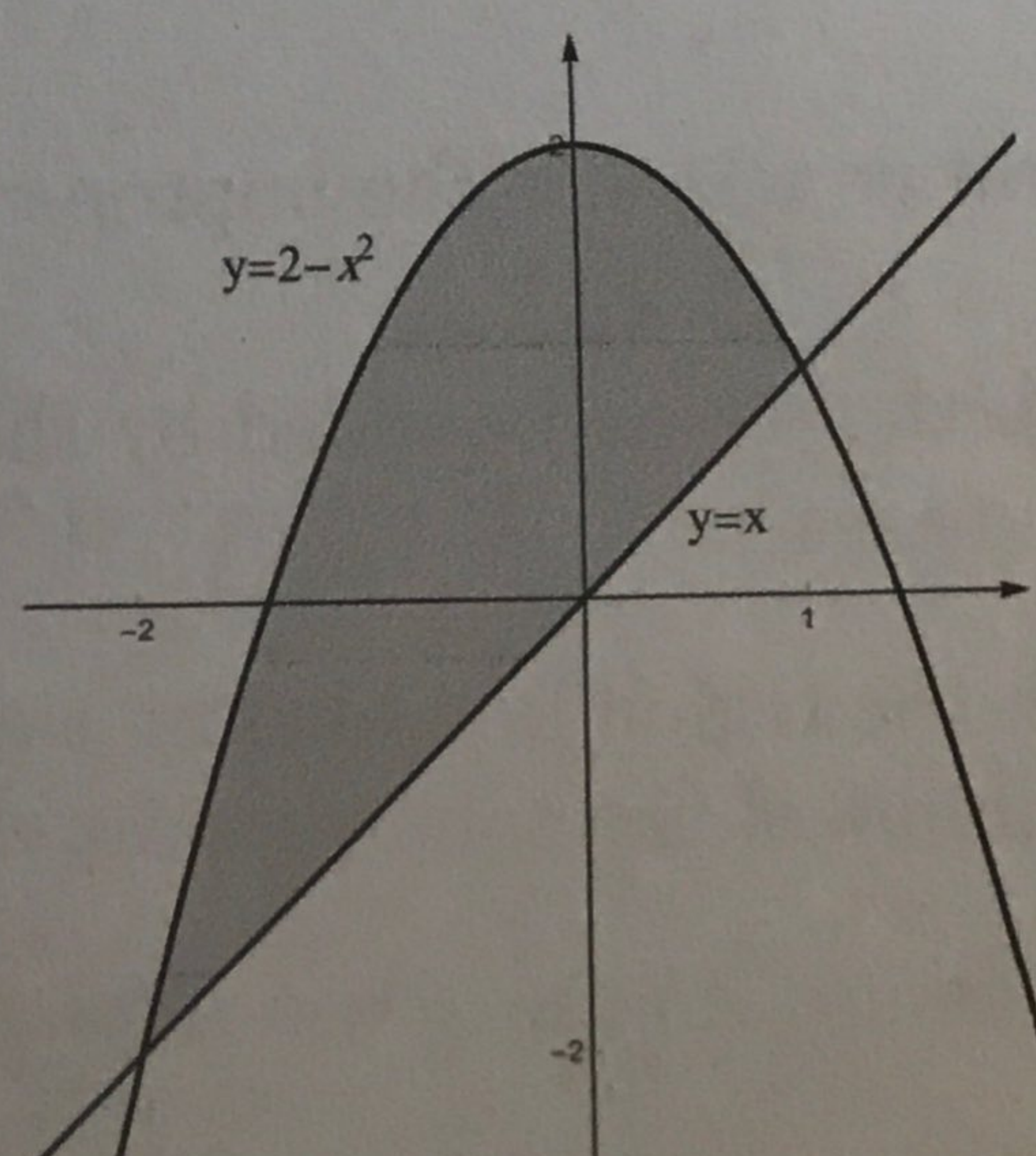
Hence, the improper integral converges.

3. Sketch the region bounded by the graphs of the curves $y = 2 - x^2$ and $y = x$, then find its area.

Solution:

$y = 2 - x^2$ represents a parabola opens downwards, and its vertex is $(2, 0)$.

$y = x$ represents a straight line passing through $(0, 0)$ and its slope is 1 .



Points of intersection of $y = 2 - x^2$ and $y = x$:

$x = 2 - x^2 \implies x^2 + x - 2 = 0$

$\implies (x+2)(x-1) = 0$

$\implies x = -2\ ,\ x = 1.$

$$A = \int_{-2}^{1} [(2 - x^2) - x] \; dx = \int_{-2}^{1} (-x^2 - x + 2) \; dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \left(-\frac{1^3}{3} - \frac{1^2}{2} + 2(1) \right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2) \right) .$$

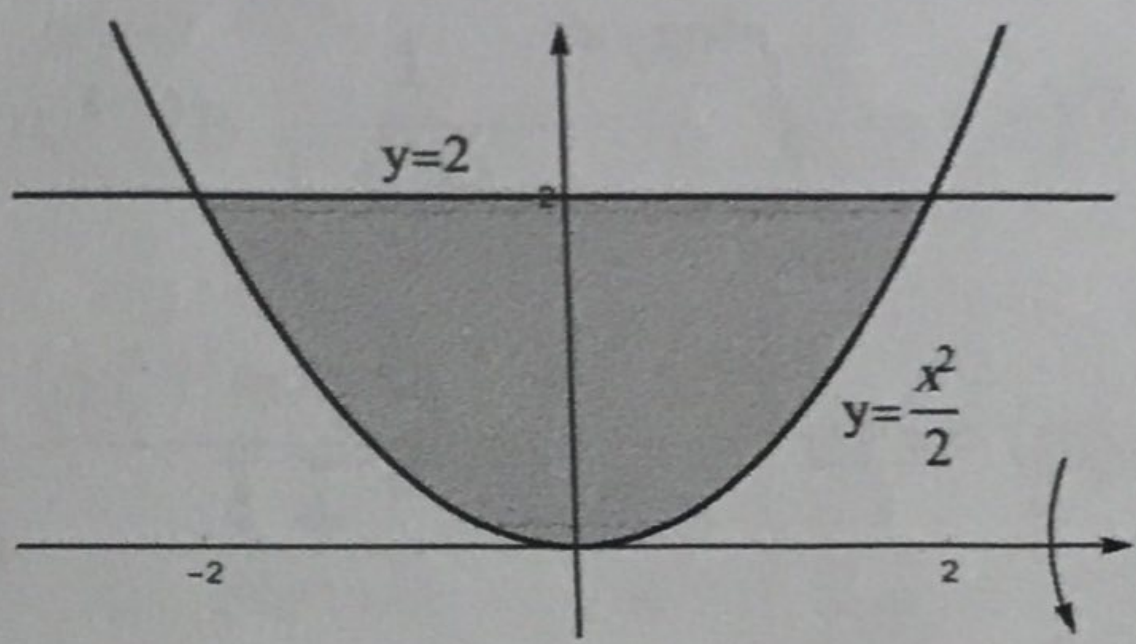
$$= -\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - 2 - 4 \right) = -\frac{9}{3} - \frac{1}{2} + 8 = 5 - \frac{1}{2} = \frac{9}{2} .$$

4. Sketch the region bounded by the graphs of the curves $y = \frac{x^2}{2}$ and $y = 2$, then find the volume of the solid generated by revolving this region about the x-axis.

Solution :

$y = \frac{x^2}{2}$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = 2$ represents a straight line parallel to the x-axis and passing through $(0, 2)$.



Points of intersection of $y = \frac{x^2}{2}$ and $y = 2$:

$$\frac{x^2}{2} = 2 \implies x^2 = 4 \implies x = \pm 2.$$

Using Washer Method :

$$V = \pi \int_{-2}^{2} \left[(2)^2 - \left(\frac{x^2}{2} \right)^2 \right] \; dx = \pi \int_{-2}^{2} \left(4 - \frac{x^4}{4} \right) \; dx$$

$$= \pi \left[4x - \frac{x^5}{20} \right]_{-2}^{2} = \pi \left[\left(4(2) - \frac{(2)^5}{20} \right) - \left(4(-2) - \frac{(-2)^5}{20} \right) \right]$$

$$= \pi \left[8 - \frac{32}{20} - \left(-8 + \frac{32}{20} \right) \right] = \pi \left(8 - \frac{8}{5} + 8 - \frac{8}{5} \right)$$

$$= \pi \left(16 - \frac{16}{5} \right) = \frac{64\pi}{5} .$$

5. Find the arc length of $y = \ln|\sec x|$, from $x = 0$ to $x = \frac{\pi}{4}$.

Solution :

$$y' = \frac{\sec x \tan x}{\sec x} = \tan x .$$

$$L = \int_{0}^{\frac{\pi}{4}} \sqrt{1 + (\tan x)^2} \; dx = \int_{0}^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \; dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} |\sec x|\, dx = \int_0^{\frac{\pi}{4}} \sec x\, dx$$

$$= [\ln|\sec x + \tan x|]_0^{\frac{\pi}{4}} = \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|\ .$$

$$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(1+\sqrt{2}\right)$$

6. Convert the polar equation $\dfrac{3}{r} = \sin\theta - \sec\theta$ into a Cartesian equation.

Solution:

$$\frac{3}{r} = \sin\theta - \sec\theta \implies 3 = r\sin\theta - r\sec\theta = r\sin\theta - \frac{r}{\cos\theta}$$

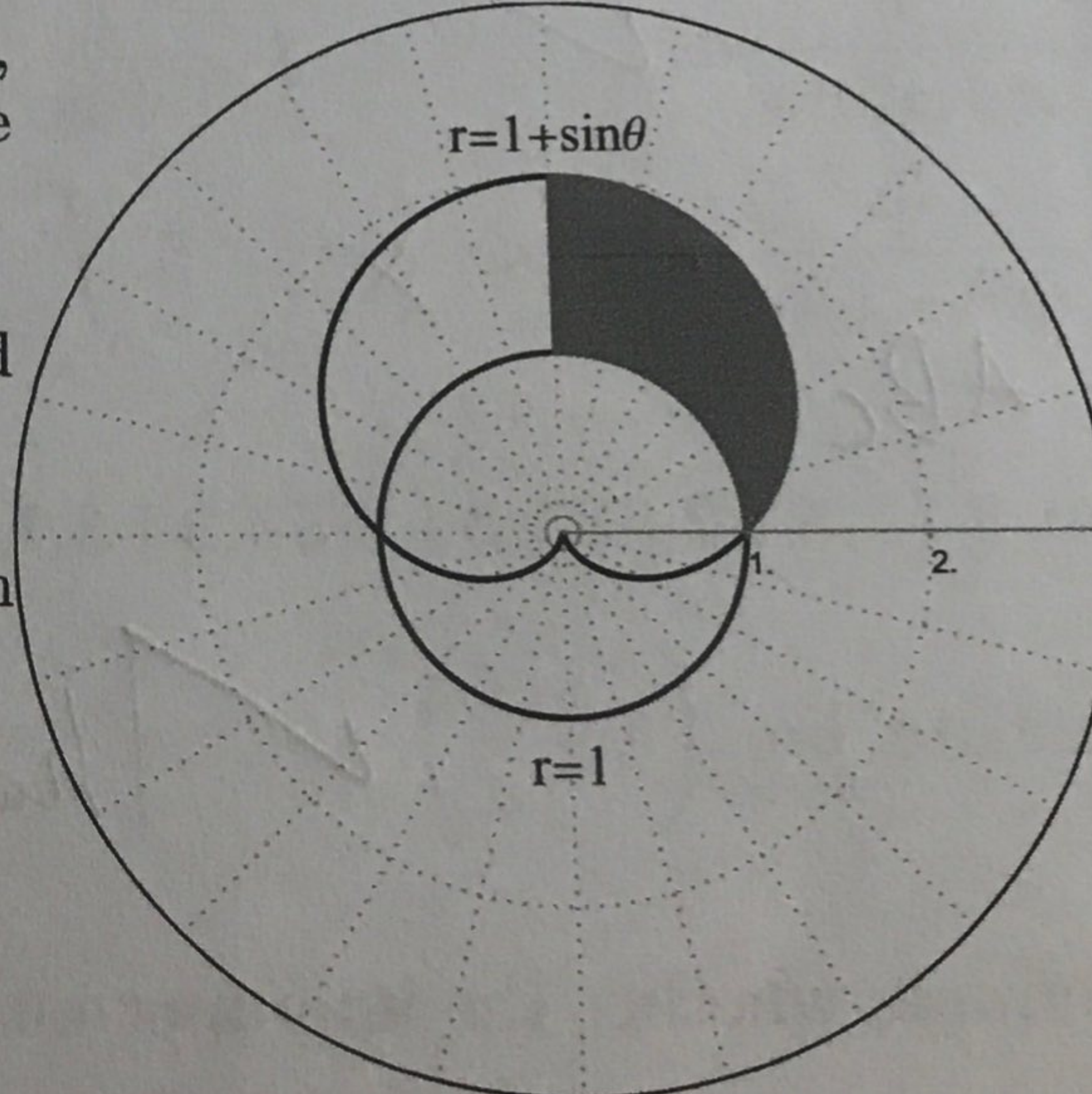
$$\implies 3 = r\sin\theta - \frac{r^2}{r\cos\theta} \implies 3 = y - \frac{x^2+y^2}{x}$$

$$\implies 3x = xy - (x^2+y^2) \implies x^2+y^2+3x-xy = 0\ .$$

7. Sketch the region in the first quadrant, inside the curve $r = 1 + \sin\theta$ and outside the curve $r = 1$, then find its area.

Solution:

$r = 1 + \sin\theta$ represents a cardioid, symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.



$r = 1$ represents a circle centered at the pole, and its radius is 1.

Points of intersection between $r = 1 + \sin\theta$ and $r = 1$:

$1 + \sin\theta = 1 \implies \sin\theta = 0$

$\implies \theta = 0\ ,\ \theta = \pi$

Since the region lies in the first quadrant then $0 \le \theta \le \dfrac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(1+\sin\theta)^2 - (1)^2\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[1 + 2\sin\theta + \sin^2\theta - 1\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[2\sin\theta + \sin^2\theta\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[2\sin\theta + \left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + 2\sin\theta - \frac{1}{4}\cos 2\theta\ (2)\right] d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\,\theta - 2\cos\theta - \frac{1}{4}\ \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{\pi}{2}\right) - \frac{1}{4}\ \sin(\pi)\right) - \left(\frac{1}{2}\,(0) - 2\cos\,(0) - \frac{1}{4}\ \sin(0)\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - 0 - 0\right) - (0 - 2 - 0)\right] = \frac{1}{2}\left(2 + \frac{\pi}{4}\right) = 1 + \frac{\pi}{8}\,.$$