Linear transformation :

Def : Let V and $V`$ be two Linear spaces and $f : V \longrightarrow V`$ be a function. f is called a <u>Linear transformation</u> from V into $V`$ if satisfies the following :

(1) $f(v_1+v_2) = f(v_1) + f(v_2)$ $\quad \forall\, v_1, v_2 \in V$

(2) $f(\lambda v) = \lambda f(v)$ $\quad \forall\, v \in V$ and $\lambda \in \mathbb{R}$

Remark : If $T : V \longrightarrow V`$ is Linear transformation, then $T(0) = 0$ because $T(0) = T(\underset{\substack{\uparrow \\ \text{scalar}}}{0} \cdot v) = 0\, T(v) = 0$

example 1 : suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$; $T(a,b) = (a,b,1)$.

It is clear that T is <u>not</u> Linear transform-

Let $v_1 = (1,1)$ $\Rightarrow$ $T(v_1) = (1,1,1)$

$v_2 = (1,3)$ $\Rightarrow$ $T(v_2) = (1,3,1)$ $\Big\} \Rightarrow T(v_1) + T(v_2) = (2,4,2)$

$v_1 + v_2 = (2,4)$ $\Rightarrow$ $T(v_1+v_2) = (2,4,1)$

So, $T(v_1+v_2) \neq T(v_1) + T(v_2)$

Example 2 : Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$; $T(a,b) = (a,b,0)$.

Then T is Linear transform because

(i) $T(v_1+v_2) = T((a,b) + (c,d))$

$= T(a+c, b+d)$

$= (a+c, b+d, 0)$ .... (1)

$T(v_1) + T(v_2) = T(a,b) + T(c,d)$

$= (a,b,0) + (c,d,0)$

$= (a+c, b+d, 0)$ .... (2)

From (1), (2)

$T(v_1+v_2) = T(v_1) + T(v_2)$

(ii) $T(\lambda v) = T(\lambda(a,b))$

$= T(\lambda a, \lambda b)$

$= (\lambda a, \lambda b = 0)$

$= \lambda(a,b,0) = \lambda T(a,b) = \lambda T(v)$

Kernal and Image of T:

Let $T: V \to V'$ be a Linear transform-. So, we can define the following:

$Ker(T) = \{ v \in V : T(v) = 0 \} \subseteq V$ (domain)

$Im(T) = \{ \nu \in V' : \nu = T(u)$ for some $u \in V \} \subseteq V'$

for example:

(1) Let $O: V \to V$ be the Zero function which is $O(\nu) = 0$. It is clear is Linear transf-

$Ker(O) = V$ and $Im(O) = 0$

(2) Let $I: V \to V$ be the indentity function which is $I(\nu) = \nu$. It is clear that it is Linear transfor-

$Ker(I) = 0$ and $Im(I) =$

(3) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T(a,b) = T(a,b,a)$

(i) show that T is Linear transfor-? it is a task.

(ii) Find $Ker\,T$ and $Im\,T$?

To find $Ker\,T$: Let $\nu = (a,b) \in Ker\,T$. Then

$(0,0,0) = \boxed{0 = T(\nu)} = T(a,b) = (a,b,a)$

So, $a = 0$ and $b = 0 \Rightarrow (a,b) = (0,0)$

Hence $Ker\,T = \{(0,0)\}$.

To find $Im\,T$: Let $\nu = (a,b,c) \in Im\,T$. Then $\exists\,(x,y)$ where $T(x,y) = (a,b,c)$

$\Rightarrow (x,y,x) = (a,b,c)$

So, $\boxed{a = c = x}$ and $\boxed{b = y}$

Hence $Im(T) = \{ (x,y,x) : x,y \in \mathbb{R} \}$

Rule 1 Let $T : V \longrightarrow V'$ be a Linear transformation.
Then $\operatorname{Ker} T \triangledown V$ (domain)

Proof (i) as $T(0) = 0$, $0 \in \operatorname{Ker} T$

(ii) Let $v_1, v_2 \in \operatorname{Ker} T$. $\boxed{\text{The goal}}$ is to Prove $v_1 + v_2 \in \operatorname{Ker} T$

$T(v_1) = 0$ $\quad T(v_2) = 0$ $\quad T(v_1 + v_2) = 0$

For that

$$\begin{aligned} L\text{-}H\text{-}S &= T(v_1 + v_2) \\ &= T(v_1) + T(v_2) = 0 + 0 = 0 = R\text{-}H\text{-}S \end{aligned}$$

$T(v) = 0$ (iii) Let $v \in \operatorname{Ker} T$ and $\lambda \in \mathbb{R}$. $\boxed{\text{The goal}}$ is to Prove that $\lambda v \in \operatorname{Ker} T$

$T(\lambda v) = 0$

For that

$$L\text{-}H\text{-}S = T(\lambda v) = \lambda T(v) = \lambda(0) = 0 = R\text{-}H\text{-}S$$

By (i), (ii) and (iii), $\operatorname{Ker}(T) \triangledown V$ ∎

Rule 2 Let $T : V \longrightarrow V'$ be a Linear Transformation.
Then $\operatorname{Im}(T) \triangledown V'$ (codomain).

Proof (i) as $T(0) = \underline{0}$, $0 \in \operatorname{Im}(T)$.

(ii) Let $v_1, v_2 \in \operatorname{Im}(T)$. Then $v_1 = T(u_1)$ and $v_2 = T(u_2)$ for some $u_1, u_2 \in V$. The goal is to Prove that $v_1 + v_2 \in \operatorname{Im}(T)$; (i.e. $v_1 + v_2 = T(\boxed{?})$)

For that : $$\begin{aligned} v_1 + v_2 &= T(u_1) + T(u_2) \\ &= T(u_1 + u_2) \end{aligned}$$

Hence $v_1 + v_2 \in \operatorname{Im}(T)$

(iii) Let $v \in \operatorname{Im}(T)$. Then $v = T(u)$ for some $u \in V$.
The goal is to Prove $\lambda v \in \operatorname{Im}(T)$ where $\lambda \in \mathbb{R}$).
For that $\lambda v = \lambda T(u) = T(\lambda u)$. Hence $\lambda v \in \operatorname{Im}(T)$

By (i), (ii), (iii) $\operatorname{Im}(T) \triangledown V'$ ∎

Remark

1. Zero Linear space has a basis ϕ, and therefore its dimension equals to zero.

2. Let $T: V_1 \to V_2$ be a Linear transformation.

(i) If $\ker T = \{0\}$ then T is called Monomorphism

(ii) If $\operatorname{Im} T = V_2$ then T is called epimorphism

(iii) If T is monomorphism and epimorphism then T is called isomorphism

Rule (3) Let $T: V_1 \to V_2$ be a Linear transformation.
If $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V_1
then $S = \{T(v_1), \dots, T(v_n)\}$ spans $\operatorname{Im}(T)$.

Proof Let $v \in \operatorname{Im}(T)$. Then $v = T(u)$ for some $u \in V_1$

Now, $u = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ (since B is basis V_1)

Therefore
$$\begin{aligned} v &= T(u) \\ &= T(\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n) \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2) + \dots + \lambda_n T(v_n) \end{aligned}$$

So, Every $v \in \operatorname{Im}(T)$ is a Linear combination of S. Hence S spans $\operatorname{Im}(T)$.

Remark : We will see later How to find a basis of $\operatorname{Im}(T)$ by Rule 3.

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Rule 5 Let $T: V_1 \to V_2$ be a Linear Transformation.
Then $\underbrace{Dim(V_1)}_{\text{Domain}} = Dim(Ker\, T) + Dim(Im\, T)$

Example Let $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T(x,y) = (x,y,x)$. It is easy to show that T is Linear transformation.

$Dim(\mathbb{R}^2) = 2$
$S_B(\mathbb{R}^2) = \{(1,0), (0,1)\}$

$T(1,0) = (1,0,1)$
$T(0,1) = (0,1,0)$

So, $\{(1,0,1), (0,1,0)\}$ spans $Im(T)$. It is clear that it is Linear indep (why?). So, $\{(1,0,1), (0,1,0)\}$ is a basis of $Im(T)$.

Now $Dim(Ker\, T) = Dim(\mathbb{R}^2) - Dim(Im(T))$
$= 2 - 2 = 0$

Hence $Ker\, T = \{0\}$ (why?) ∎

Example : Let $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T(x,y) = (x,y,0)$.
$Dim(\mathbb{R}^2) = 2$ $S_B(\mathbb{R}^2) = \{(1,0), (0,1)\}$
$T(1,0) = (1,0,0)$; $T(0,1) = (0,1,0)$.
Hence, $S = \{(1,0,0), (0,1,0)\}$ spans $Im(T)$ and S is Linearly indep $\Rightarrow$ S is a basis of $Im(T)$

Now $Dim\, Ker(T) = Dim(\mathbb{R}^2) - Dim(Im\, T)$
$= 2 - 2 = 0 \Rightarrow Ker(T) = \{0\}$ ∎

Example Let $T: \mathbb{R}^2 \to P_2(x)$; $T(a,b) = ax + bx^2$

(i) show that T is Linear transformation. (Task)

(ii) Find $\mathrm{Dim}(\ker T)$, $\mathrm{Dim}(\mathrm{Im}\, T)$.

Sol (ii) $\mathrm{Dim}(\mathbb{R}^2) = 2$, $S_B(\mathbb{R}^2) = \{(1,0), (0,1)\}$

$T(1,0) = x$; $T(0,1) = x^2$

So, $S = \{x, x^2\}$ spans $\mathrm{Im}(T)$ and Linearly indep $\Rightarrow$ S is a basis of $\mathrm{Im}(T)$

$\Rightarrow \mathrm{Dim}(\mathrm{Im}(T)) = 2$

Now, $\mathrm{Dim}(\ker T) = \mathrm{Dim}(\mathbb{R}^2) - \mathrm{Dim}(\mathrm{Im}(T)) = 2-2 = 0$ ☒

Excersice: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transformation where $T(1,1) = (1,2)$ and $T(2,1) = (4,7)$.

Find $T(5,7)$?

Sol Let $S = \{(1,1), (2,1)\}$. Then S is a basis of $\mathbb{R}^2$ (why).

There fore, $(5,7)$ is a Linear combination of S. So, $(5,7) = \lambda_1 (1,1) + \lambda_2 (2,1)$

$$\Rightarrow \left.\begin{array}{l} \lambda_1 + 2\lambda_2 = 5 \\ \lambda_1 + \lambda_2 = 7 \end{array}\right\} \Rightarrow \lambda_1 = 3 \text{ and } \lambda_2 = 2$$

Hence $(5,7) = 3(1,1) + 2(2,1)$

$$\Rightarrow T(5,7) = T(3(1,1) + 2(2,1))$$
$$= 3\,T(1,1) + 2\,T(2,1)$$
$$= 3(1,2) + 2(4,7)$$
$$= (3,6) + (8,14) = (9,20)$$ ☒

Matrix of Linear transformation

Case 1 $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is Linear transformation

$B_1 = \{v_1, \ldots, v_n\}$ $\quad$ $B_2 = \{u_1, \ldots, u_m\}$

Then $M = T_{B_1}^{B_2}$

$$= \begin{bmatrix} [T(v_1)]_{B_2} & [T(v_2)]_{B_2} & \cdots & [T(v_n)]_{B_2} \end{bmatrix}$$

is called matrix of Linear transformation

in this case $\boxed{[v]_{B_1} \cdot T_{B_1}^{B_2} = [T(v)]_{B_2}}$

example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$; $(x, y) = (x, 0)$

$B_1 =$ Standered basis $\quad$ $B_2 = \{(1,1), (1,2)\}$

Now $T(1,0) = (1,0)$ & $T(0,1) = (0,0)$

suppose $T(1,0) = \lambda_1(1,1) + \lambda_2(1,2)$

$\Rightarrow (1,0) = \lambda_1(1,1) + \lambda_2(1,2)$

$\Rightarrow \lambda_1 + \lambda_2 = 1$

$\lambda_1 + 2\lambda_2 = 0$

$\Rightarrow \lambda_2 = -1$

$\lambda_1 = 2$

$\Rightarrow [T(1,0)]_{B_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$T(0,1) = \lambda_1(1,1) + \lambda_2(1,2)$

$\Rightarrow (0,0) = \lambda_1(1,1) + \lambda_2(1,2)$

$\Rightarrow \lambda_1 + \lambda_2 = 0$

$\lambda_1 + 2\lambda_2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 0$

$\Rightarrow [T(0,1)]_{B_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence, $M = T_{B_1}^{B_2} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$

** suppose $[v]_{B_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$T_{B_1}^{B_2}[v]_{B_1} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} = [T(v)]_{B_2}$$

$\Rightarrow T(v) = -1(1,1) + 4(1,2) = (3,7)$

Example Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a Linear transformation with matrix $M = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$

find $T(3,5)$. Consider the basis are the standened basis ?

Sol $\left[T(1,0)\right]_{B_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow T(1,0) = 1(1,0,0) + 2(0,1,0) + (-1)(0,0,1) = (1,2,-1)$

$\left[T(0,1)\right]_{B_2} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \Rightarrow T(0,1) = (3,2,-1)$

Now

$(3,5) = 3(1,0) + 5(0,1)$

So, $T(3,5) = T(3(1,0) + 5(0,1))$
$= 3T(1,0) + 5T(0,1)$
$= 3(1,2,-1) + 5(3,2,-1) = (18,16,-8)$ ■

Remark By above, if we Know basis and the matrix of Linear transformation, we can Compute the image of any vector.

★★ **SOME Rules for this case :**

Rule $\left.\begin{array}{l} T_1: \mathbb{R}^n \to \mathbb{R}^m \text{ Linear transf-} \\ T_2: \mathbb{R}^m \to \mathbb{R}^k \text{ Linear transf-} \end{array}\right\} \Rightarrow T = T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^m$ with standened matrix $A = \underbrace{A_2}_{\text{s.M of } T_2} \underbrace{A_1}_{\text{s.M of } T_1}$

Rule $T: \mathbb{R}^n \to \mathbb{R}^n$ isomorphism $\Leftrightarrow$ A (standerd matrix of T) is invertible

Definition Let $T: V_1 \to V_2$ be Linear transformation. Then
Nullity $(T) = \text{Dim}(\text{Ker } T)$
Rank $(T) = \text{Dim}(\text{Im } T)$

Rule Nullity (T) + Rank(T) = Dim V_1 (Domain)

Example Let $T: P_2(x) \to P_1(x)$; $f(x) = f'(x)$

(1) show that T is Linear transfor- ?

(2) find Nullity of T and Rank of T ?

(3) find the matrix of Linear transfor T ?

Sol (1) (i) $T(f_1 + f_1) = (f_1 + f_2)' = f_1' + f_2' = T(f_1) + T(f_2)$

(ii) $T(\lambda f) = (\lambda f)' = \lambda f' = \lambda T(f)$

(2) $S_B(P_2(x)) = \{1, x, x^2\}$

$T(1) = 0$ $\quad T(x) = 1$ $\quad T(x^2) = 2x$

so $S = \{0, 1, 2x\}$ spans $Im(T) \Rightarrow \{1, 2x\}$ basis $Im(T)$

So, $Dim\ Im(T) = 2 = Dim(P_1(x)) \Rightarrow \boxed{Im(T) = P_1(x)}$

$Dim(Ker\ T) = Dim\ P_2(x) - Dim\ Im(T) = 3 - 2 = 1$

Hence $Nullity(T) = 1$ $\quad Rank(T) = 2$

(3) Consider $S_{B_1}(P_2(x)) = \{1, x, x^2\}$

$S_{B_2}(P_1(x)) = \{1, x\}$

$T(1) = 0 \Rightarrow [T(1)]_{S_{B_2}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $0 = 0(1) + 0(x)$

$T(x) = 1 \Rightarrow [T(x)]_{S_{B_2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $1 = 1(1) + 0(x)$

$T(x^2) = 2x \Rightarrow [T(x^2)]_{S_{B_2}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ for $2x = 0(1) + 2(x)$

Hence $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Example Let $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x + 2y, 0, 3z)$

(i) Find $Rank(T)$ and nullity of T ?

(ii) find the standened matrix of T ?

Sol (ii) $S_B(\mathbb{R}^3) = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$\Rightarrow [T(1,0,0)]_{S_B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(0,1,0) = (2,0,0) \Rightarrow [T(0,1,0)]_{S_B} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(0,0,1) = (0,0,3) \Rightarrow [T(0,0,1)]_{S_B} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(i) $T(1,0,0) = (1,0,0)$ $\quad T(0,1,0) = (2,0,0)$ $\quad T(0,0,1) = (0,0,3)$ $\boxed{10}$

$\Rightarrow \quad S = \{\underbrace{(1,0,0), (2,0,0)}_{\text{Linear dep}}, (0,0,3)\}$ spans $Im(T)$

$\Rightarrow \quad S' = \{(1,0,0), (0,0,3)\}$ basis of $Im(T)$

$\Rightarrow \quad Rank(T) = Dim(Im(T)) = 2$

$\Rightarrow \quad$ Nullity of $(T) = 3 - Rank(T) = 1$

another sol

To Find $Ker(T)$:

$Ker(T) = \{(x,y,z) : (x+2y, 0, 3z) = (0,0,0)\}$

$\Rightarrow \quad \left.\begin{array}{r} x+2y = 0 \\ 3z = 0 \end{array}\right\}$

$\Rightarrow \quad z = 0 \qquad x = t$ and $y = \frac{-t}{2}$

so, $Ker(T) = \{(t, \frac{-t}{2}, 0) \; ; t \in \mathbb{R}\}$

$= \{(1, \frac{-1}{2}, 0)t \; ; t \in \mathbb{R}\}$

Hence basis of $Ker(T) = \{(1, \frac{-1}{2}, 0)\}$

$\Rightarrow$ Nullity of $(T) = 1$

$\Rightarrow$ Rank of $(T) = 3 - 1 = 2$

Ex Let $T: \mathbb{R}^3 \to \mathbb{R}$; $T(x,y,z) = \langle (x,y,z), (1,0,-1) \rangle$

(1) Prove T is Linear transfor ?

(2) Find $\ker T$ and $\operatorname{Im} T$? Find $\operatorname{Rank}(T)$, nullity (T) ?

(3) Find the matrix of T ?

Sol **(1)** (i) $T(v_1+v_2) = \langle v_1+v_2, (1,0,-1) \rangle$
$= \langle v_1, (1,0,-1) \rangle + \langle v_2, (1,0,-1) \rangle$
$= T(v_1) + T(v_2)$

(ii) $T(\lambda v) = \langle \lambda v, (1,0,-1) \rangle = \lambda \langle v, (1,0,-1) \rangle$
$= \lambda T(v)$

(2) $B_{\mathbb{R}^3} = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = \langle (1,0,0), (1,0,-1) \rangle = 1$

$T(0,1,0) = \langle (0,1,0), (1,0,-1) \rangle = 0$

$T(0,0,1) = \langle (0,0,1), (1,0,-1) \rangle = -1$

So, $S = \{1, 0, -1\}$ spans $\operatorname{Im}(T) \Rightarrow$ $\boxed{\{1\} \text{ basis of } \operatorname{Im}(T)}$

To Find Ker T

$\ker(T) = \{(x,y,z) : \langle (x,y,z), (1,0,-1) \rangle = 0\}$
$= \{(x,y,z) : x - z = 0\}$

Let $x = t$, $z = t$

So, $\ker(T) = \{(t, y, t) : t, y \in \mathbb{R}\}$
$= \{(t,0,t) + (0,y,0) : t, y \in \mathbb{R}\}$
$= \{t(1,0,1) + y(0,1,0)\}$

$B_{\ker(T)} = \{(1,0,1), (0,1,0)\}$

Nullity $(T) = 2$

$\Rightarrow \operatorname{Rank}(T) = 1$

(3) $A = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

Do Not forget

$$T : V \longrightarrow W$$

V: basis B $\qquad$ W: basis C

★ the matrix of $T = T_B^C = {}_C T_B = \underbrace{\left[[T(u_1)]_C \cdots [T(u_n)]_C \right]}_{\text{Coordinate of Image of B respect to C}}$

★ $T_B^C \cdot [v]_B = [T(v)]_C$ for any $v \in V$.

Ex Let $T : \mathbb{R}^2 \longrightarrow P_1(x)$; $T(a,b) = ax$. Find T_B^C ?

Sol $B = \{(1,0), (0,1)\}$ $\qquad$ $C = \{1, x\}$

$T(1,0) = x = 0(1) + 1(x) \Rightarrow [T(1,0)]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(0,1) = 0 = 0(1) + 0(x) \Rightarrow [T(0,1)]_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow T_B^C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

special case

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ where B_1, B_2 be two basis

we have the following :

T_{B_1} = standered matrix of B_1

T_{B_2} : standered matrix of B_2

$P_{B_2 B_1}$ = trasition matrix from B into C

In this case

$$\boxed{T_{B_2} = P_{B_2 B_1} \, T_{B_1} \, P_{B_1 B_2}} \longrightarrow \text{Rule}$$

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transformation where $B = \{(-1,0), (1,2)\}$

$C = \{(1,0), (0,1)\}$ be two basis

If $T_C = \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix}$; find T_B

Sol

we have to find ${}_C P_B$:

Let $(-1,0) = \lambda_1 (1,0) + \lambda_2 (0,1)$

$\Rightarrow \boxed{\lambda_1 = -1 \quad \wedge \quad \lambda_2 = 0}$

Let $(1,2) = \lambda_1 (1,0) + \lambda_2 (0,1)$

$\Rightarrow \lambda_1 = 1 \quad \wedge \quad \lambda_2 = 2$

So ${}_C P_B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

${}_B P_C = ({}_C P_B)^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$

Hence $T_B = {}_B P_C \; T_C \; {}_C P_B$

$= \begin{bmatrix} -1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

(Ex) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standered matrix

$A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & 3 \\ 3 & 3 & -1 \end{bmatrix}$. find $T(1,2,5)$?

$T(1,0,0)$

$T(0,1,0)$

Sol $(1,2,5) = 1(1,0,0) + 2(0,1,0) + 5(0,0,1)$

$\Rightarrow T(1,2,5) = 1\, T(1,0,0) + 2\, T(0,1,0) + 5\, T(0,0,1)$

$= \ldots\ldots\ldots$