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Final Exam	Math 111	May 2025

Question 1 : (3+2+2)

1. Find the value of C that satisfies the mean value theorem for the definite integral of the function $f(x) = \sqrt[3]{x-1}$ on the interval $[1, 9]$.
2. Find $\dfrac{dF(x)}{dx}$ if $F(x) = \displaystyle\int_{\cos x}^{2^{3x}} \sqrt{1+t^3}dt$.
3. Find $\dfrac{dy}{dx}$ if $y = \tanh(\sqrt{x}) + \sinh^{-1}(\dfrac{2}{\sqrt{x}})$.

Question 2 : (3+3+3+3+2)
Compute the following integrals:

1. $\displaystyle\int \frac{dx}{\sqrt{-x^2-2x+8}}$;
2. $\displaystyle\int x \tanh^{-1}(x) dx$;
3. $\displaystyle\int \frac{3x^2+x+8}{x^3+4x} dx$;
4. $\displaystyle\int \frac{dx}{(9+x^2)^{\frac{3}{2}}}$
5. $\displaystyle\int \frac{dx}{(4.2^{-x}+2^x)}$;

Question 3 : (2+3+3+3+3+1+4)

1. Evaluate the following limit $\displaystyle\lim_{x\to+\infty} \frac{x - xe^x}{x + 2xe^x}$.
2. Discuss whether the improper integral $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2+4}$ converges or diverges.
3. Sketch the region bounded by the curves $y = 5x^2$ and $y = 4 + x^2$ and find its area.
4. Sketch the region bounded by the curves $y = x^2$, $y = -x + 2$ and find the volume of the solid generated by the revolving this region about the $x-$axis.
5. Find the arc length of the curve defined by: $y = \dfrac{e^x + e^{-x}}{2}$. on the interval $[0, \ln 3]$.
6. Convert the polar equation $\dfrac{1}{r} = \dfrac{\sin\theta + 5\cos\theta}{2}$ into Cartesian equation.
7. Sketch the region inside $r = 1 + \cos\theta$ that in the second quadrant. Then compute its area

Final exam Math 111 – S II

Q1:

1) $(b-a)\,f(c) = \int_a^b f(x)\,dx$ ✓1

$8\sqrt[3]{c-1} = \frac{3}{4}\left[(x-1)^{\frac{4}{3}}\right]_1^9 = 12$ ✓1

$\sqrt[3]{c-1} = \frac{3}{2} \Rightarrow c-1 = \frac{27}{8}$ ✓1

$\Rightarrow c = \frac{35}{8} \in (1,9)$

2) $F(x) = \int_{\cos x}^{2^{3x}} \sqrt{1+t^3}\,dt$

$\frac{dF(x)}{dx} = \sqrt{1+2^{9x}}\,(2^{3x}\cdot 3\ln 2) + \sqrt{1+\cos^3 x}\,\sin x$ ✓1 ✓1

3) $y = \tanh(\sqrt{x}) + \sinh^{-1}\left(\frac{2}{\sqrt{x}}\right)$

$y' = \frac{1}{2\sqrt{x}}\operatorname{sech}^2(\sqrt{x}) + \frac{-x^{-\frac{3}{2}}}{\sqrt{1+\frac{4}{x}}}$ 1 ✓1

Q2:

1. $\int \frac{dx}{\sqrt{-x^2-2x+8}} = \int \frac{dx}{\sqrt{9-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{3}\right) + C$ ✓2 ✓1

2. $I = \int x\tanh^{-1}x\,dx$

$\tanh^{-1}x = u \Rightarrow du = \frac{dx}{1-x^2}$

$x\,dx = dv \Rightarrow v = \frac{x^2}{2}$ ✓1

$I = \frac{x^2}{2}\tanh^{-1}x - \frac{1}{2}\int \frac{x^2}{1-x^2}\,dx$

$$I = \frac{x^2}{2}\tanh^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1-x^2}\,dx$$

$$= \frac{x^2}{2}\tanh^{-1}x - \frac{1}{2}\int \left(-1 + \frac{1}{1-x^2}\right) dx \quad 1$$

$$= \frac{x^2}{2}\tanh^{-1}x - \frac{1}{2}\left(-x + \tanh^{-1}x\right) + C \quad 1$$

3. $I = \int \frac{3x^2+x+8}{x^3+4x}\,dx$

$$\frac{3x^2+x+8}{x^3+4x} = \frac{3x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$3x^2+x+8 = A(x^2+4) + Bx^2 + Cx$$

$$3x^2+x+8 = (A+B)x^2 + Cx + 4A \Rightarrow \quad 1.5$$

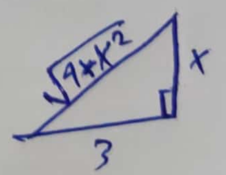
$$\left.\begin{array}{l} A+B=3 \\ C=1 \\ 4A=8 \end{array}\right\} \Rightarrow A=2,\ B=1,\ C=1$$

$$I = \int \frac{2}{x}\,dx + \int \frac{x+1}{x^2+4}\,dx = \int \frac{2}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{dx}{4+x^2}$$

$$= 2\ln|x| + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C \quad 1.5$$

4. $I = \int \frac{dx}{(9+x^2)^{\frac{3}{2}}}$

$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3}$ 1



$dx = 3\sec^2\theta\, d\theta$

$$I = \int \frac{3\sec^2\theta\, d\theta}{(9+9\tan^2\theta)^{\frac{3}{2}}} = \frac{1}{9}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{9}\int \cos\theta\, d\theta \quad 1$$

$$I = \frac{1}{9}\sin\theta + c = \frac{1}{9}\frac{x}{\sqrt{9+x^2}} + c \quad 1$$

$$5. \quad I = \int \frac{dx}{(4 \cdot 2^{-x} + 2^x)} = \int \frac{2^x}{(4+2^{2x})}dx = \int \frac{2^x}{(4+(2^x)^2)}dx \quad 1$$

$$I = \frac{1}{2\ln 2}\tan^{-1}\left(\frac{2^x}{2}\right) + C \quad 1$$

$\underline{\underline{Q_3}}$

$$1. \quad \lim_{x \to +\infty} \frac{x - xe^x}{x + 2xe^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to +\infty} \frac{-\cancel{e^x}(2+x)}{\cancel{e^x}(4+2x)} \quad 1 \quad = \lim_{x \to \infty} \frac{-(x+2)}{2x+4} = -\frac{1}{2} \quad 1$$

$$2. \quad \int_{-\infty}^{+\infty} \frac{dx}{4+x^2} = \int_{-\infty}^{0} \frac{dx}{4+x^2} + \int_{0}^{+\infty} \frac{dx}{4+x^2}$$

$$= \lim_{t \to -\infty} \int_t^0 \frac{dx}{4+x^2} + \lim_{c \to +\infty} \int_0^c \frac{dx}{4+x^2} \quad 1$$

$$= \lim_{t \to -\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_t^0 + \lim_{c \to +\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^c \quad 1$$

$$= \lim_{t \to -\infty} \left[\frac{1}{2}\left(\tan^{-1}(0) - \tan^{-1}\left(\frac{t}{2}\right)\right)\right]$$

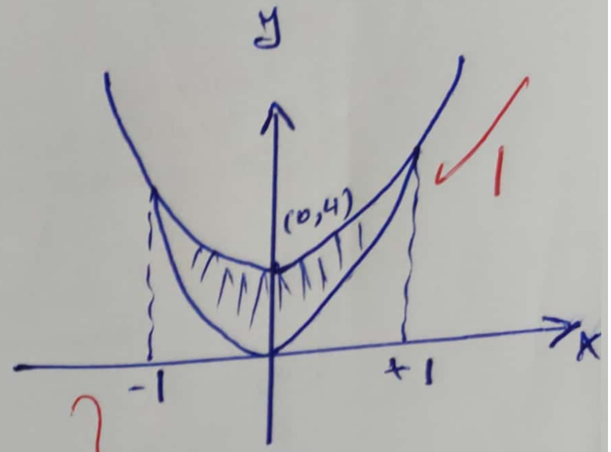
$$+ \lim_{c \to +\infty} \left[\frac{1}{2}\left(\tan^{-1}\left(\frac{c}{2}\right) - \tan^{-1}(0)\right)\right] \quad 1$$

$$= \frac{1}{2}\left(0 + \frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2} \quad \text{c'gt}$$

3. $y = 5x^2$ & $y = 4 + x^2$

$5x^2 = 4 + x^2 \Rightarrow 4x^2 = 4 \Rightarrow x^2 = 1$

$(x-1)(x+1) = 0 \Rightarrow x = -1, x = +1$

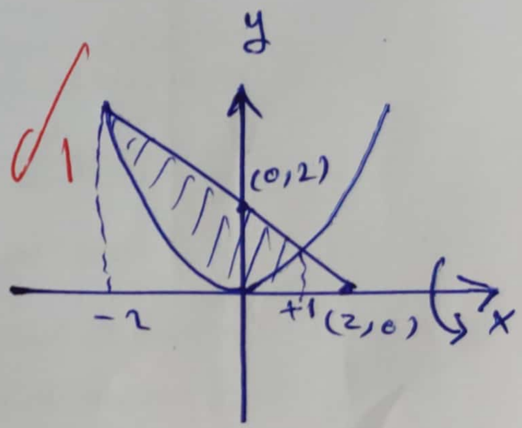


$$A = \int_{-1}^{1} [(4 + x^2) - 5x^2]\, dx$$

$$A = \int_{-1}^{+1} [4 - 4x^2]\, dx = \left[4x - \frac{4}{3}x^3\right]_{-1}^{+1}$$

$$= \left(4 - \frac{4}{3}\right) - \left(-4 + \frac{4}{3}\right) = \frac{16}{3}$$

4. $y = x^2$ & $y = -x + 2$



$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \Rightarrow x = 1, x = -2$

$$V = \pi \int_{-2}^{1} [(2-x)^2 - (x^2)^2]\, dx$$

$$= \pi \int_{-2}^{1} [4 - 4x + x^2 - x^4]\, dx$$

$$= \pi \left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{1}$$

$$\pi \left[\left(4 - 2 + \frac{1}{3} - \frac{1}{5}\right) - \left(-8 - 8 - \frac{8}{3} + \frac{2^5}{5}\right)\right]$$

5. $y = \frac{e^{x}+e^{-x}}{2} = \cosh x$ & $[0, \ln 3]$

$y' = \sinh(x)$

$L = \int_0^{\ln 3} \sqrt{1+(y')^2}\,dx = \int_0^{\ln 3} \sqrt{1+\sinh^2 x}\,dx = \int_0^{\ln 3} \cosh x\,dx$ ✓1

$= \left[\sinh x\right]_0^{\ln 3} = \frac{1}{2}\left[e^{x} - e^{-x}\right]_0^{\ln 3} =$ ✓1 ✓1

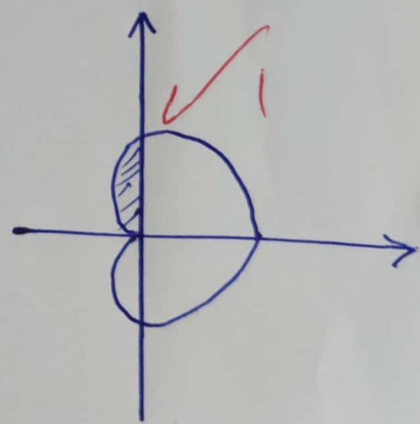
$= \frac{1}{2}\left[3 - \frac{1}{3}\right] = \frac{\frac{8}{3}}{2} = \frac{4}{3}$

6. $\frac{1}{r} = \frac{\sin\theta + 5\cos\theta}{2} \Rightarrow 2 = r\sin\theta + 5r\cos\theta$ ✓0.5

$2 = y + 5x \Rightarrow y = -5x + 2$ ✓0.5

7. $r = 1+\cos\theta$

$\theta = \frac{\pi}{2}$



$\frac{\pi}{2} \leq \theta \leq \pi$ ✓1

$A = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\,d\theta$ ✓1

$= \frac{1}{2}\int_{\pi/2}^{\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$ ✓1

$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$ ✓1

$= \frac{1}{2}\left[\left(\frac{3\pi}{2}+0+0\right) - \left(\frac{3\pi}{4}+2+0\right)\right] = \frac{3\pi}{8}$ -1