

Chapter 5: Bayesian Estimation

- 5.1: Let $X_1, X_2, \dots, X_n$ be a random sample from Bernoulli with parameter θ , and the prior distribution of θ is a uniform distribution, where $0 < \theta < 1$. Find the posterior distribution and the Bayes' point estimator of θ when the loss function be the squared error loss function

given that $X_i \sim \text{Bernoulli}(\theta) \Rightarrow f(x|\theta) = \theta^x(1-\theta)^{1-x}, \quad x = 0, 1$

$$\theta \sim \text{Uniform}(0, 1) \Rightarrow h(\theta) = 1, \quad 0 < \theta < 1$$

$$L(x|\theta) = \prod_{i=1}^n f(x_i|\theta) = \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i}$$

- (a) The posterior distribution:

$$k(\theta|x) \propto L(x|\theta)h(\theta)$$

$$K(\theta|x) \propto \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i} = \theta^{\sum_{i=1}^n x_i + 1 - 1} (1-\theta)^{n-\sum_{i=1}^n x_i + 1 - 1}$$

Then $\theta|X \sim \text{Beta}(\sum_{i=1}^n x_i + 1, n - \sum_{i=1}^n x_i + 1)$

- (b) Then the Bayes Point estimator of θ is

$$E(\theta|X) = \frac{\sum_{i=1}^n x_i + 1}{\sum_{i=1}^n x_i + 1 + n - \sum_{i=1}^n x_i + 1} = \frac{\sum_{i=1}^n x_i + 1}{n+2}$$

- The pdf of Beta Distribution
 $X \sim \text{Beta}(a, b)$

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} [1-x]^{b-1}$$

$$E(X) = \frac{a}{a+b}$$

- 5.2: Let Y have a binominal distribution in which $n = 20$ and $p = \theta$. The prior probability on θ is $\text{Beta}(a, b)$, where $a, b > 0$ are known constants. Find the following:

- (a) Posterior distribution.

- (b) Bayes' point estimate of θ , when $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$.

given that $Y \sim \text{Binomial}(20, \theta) \Rightarrow f(y|\theta) = \binom{20}{y} \theta^y (1-\theta)^{20-y}, \quad y = 0, 1, \dots, 20$

$$\theta \sim \text{Beta}(a, b), a, b > 0 \Rightarrow h(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} [1-\theta]^{b-1}, \quad 0 < \theta < 1$$

$$L(y|\theta) = \prod_{i=1}^{20} f(y_i|\theta) = \prod_{i=1}^{20} \binom{20}{y_i} \theta^{\sum_{i=1}^{20} y_i} (1-\theta)^{400 - \sum_{i=1}^{20} y_i}$$

(a) The posterior distribution

$$K(\theta|y) \propto \theta^{\sum_{i=1}^{20} y_i} (1-\theta)^{400-\sum_{i=1}^{20} y_i} \theta^{a-1} [1-\theta]^{b-1}$$

$$\propto \theta^{\sum_{i=1}^{20} y_i + a - 1} (1-\theta)^{400 - \sum_{i=1}^{20} y_i + b - 1}$$

$$\text{Then } \theta|Y \sim \text{Beta}(\sum_{i=1}^{20} y_i + a, 400 - \sum_{i=1}^{20} y_i + b)$$

(b) the Bayes Point estimator of θ

$$E(\theta|X) = \frac{\sum_{i=1}^{20} y_i + a}{400 + a + b}$$

5.3: Let $X_1, X_2, \dots, X_{10}$ denote a random sample from a Poisson distribution with mean θ ,

$0 < \theta < \infty$. Let $Y = \sum_{i=1}^{10} X_i$. Use the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf $h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$, for $0 < \theta < \infty$. Find:

(a) The posterior distribution.

(b) The Bayes' solution $\delta(y)$ for a point estimate for θ , when $Y = 22$.

given that $X_1, X_2, \dots, X_{10} \sim \text{poisson}(\theta)$, $0 < x < \infty \Rightarrow f(x|\theta) = \frac{e^{-\theta} \theta^x}{x!}$

$$Y = \sum_{i=1}^{10} X_i \quad h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16} \text{ for } 0 < \theta < \infty$$

(a) The posterior distribution

$$L(x|\theta) = \prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^{10} \frac{e^{-\theta} \theta^{x_i}}{x_i!} = \frac{e^{-10\theta} \theta^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x_i!}$$

$$k(\theta|x) \propto L(x|\theta)h(\theta)$$

$$K(\theta|X) \propto e^{-10\theta} \theta^{\sum_{i=1}^n x_i} \theta^2 e^{-\frac{1}{2}\theta} = \theta^{\sum_{i=1}^n x_i + 2} e^{-10.5\theta} = \theta^{\sum_{i=1}^n x_i + 2 + 1 - 1} e^{-10.5\theta}$$

$$\text{Then } \theta|X \sim \text{Gamma}(\sum_{i=1}^n x_i + 3, \frac{1}{10.5})$$

- The pdf of Gamma Distribution

$$X \sim \text{Gamma}\left(n, \frac{1}{a}\right)$$

$$f(x) = \frac{a^n}{\Gamma(n)} x^{n-1} e^{-ax}$$

(b) $Y = 22$ the Bayes Point estimator of θ is

$$E(\theta|X) = \frac{\sum_{i=1}^{10} x_i + 3}{10.5} = \frac{22+3}{10.5} = 2.38, (Y = \sum_{i=1}^{10} X_i = 22)$$

(c) 90% CI for θ $Y = 22$

$$\text{Note: } \text{Gamma}(\underline{a}, \theta = \underline{2}) \equiv \chi^2_v \Rightarrow v = 2a$$

We have from (a) $\theta|X \sim \text{Gamma}(\sum_{i=1}^n x_i + 3, \frac{4}{10.5})$

$$\Rightarrow 2(10.5)\theta|X \sim \text{Gamma}(\sum_{i=1}^n x_i + 3, 2)$$

$$X \sim \text{Gamma}(n, \theta) \Rightarrow aX \sim \text{Gamma}(n, a\theta)$$

$$2(10.5)\theta|X \sim \text{Gamma}(\sum_{i=1}^n x_i + 3, 2) \equiv 2(10.5)\theta|X \sim \chi^2_{2(\sum_{i=1}^n x_i + 3)}$$

$$\Rightarrow 21\theta|X \sim \chi^2_{2(22+3)}$$

$$\Rightarrow 21\theta|X \sim \chi^2_{50}$$

Then, $\chi^2_{1-\frac{\alpha}{2}, 50} < 21\theta < \chi^2_{\frac{\alpha}{2}, 50}$

$$\alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow 1 - \frac{\alpha}{2} = 0.95 \Rightarrow \chi^2_{0.95, 50} = 34.76 \quad \chi^2_{0.05, 50} = 67.5$$

$$34.76 < 21\theta < 67.5$$

$$1.66 < \theta < 3.21$$

5.4: Let Y be the n th order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta}, 0 < x \leq \theta$, zero elsewhere. Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. Let θ be an observed value of the random variable Θ , which has pdf $h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y)$ for a point estimate of θ .

given that $f(x) = \frac{1}{\theta}, 0 < x < \theta$

$$h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty, \alpha > 0, \beta > 0$$

$$Y_n = \max(X_1, X_2, \dots, X_n)$$

$$f_{Y_n}(y_n) = n f_X(y_n) [F_X(y_n)]^{n-1}, \quad 0 \leq y_n < \theta$$

$$= n \frac{1}{\theta} \left[\frac{y_n}{\theta} \right]^{n-1} = \frac{n y_n^{n-1}}{\theta^n}$$

$$K(\theta|y_n) \propto \frac{1}{\theta^n \theta^{\beta+1}}$$

$$K(\theta|y_n) = c \frac{1}{\theta^{n+\beta+1}}, \quad \alpha < \theta < \infty$$

To find the constant c :

Since $K(\theta|y_n)$ is a pdf , then

$$\int_{\alpha}^{\infty} K(\theta|y_n) d\theta = 1$$

$$\Rightarrow \int_{\alpha}^{\infty} c \frac{1}{\theta^{n+\beta+1}} d\theta = 1$$

$$\Rightarrow \int_{\alpha}^{\infty} c \theta^{-n-\beta-1} d\theta = 1$$

$$c \left[\frac{\theta^{-n-\beta-1+1}}{-n-\beta-1+1} \right]_{\alpha}^{\infty} = 1 \Rightarrow c \left[\frac{\theta^{-n-\beta}}{-n-\beta} \right]_{\alpha}^{\infty} = 1$$

$$\Rightarrow \frac{c}{-n-\beta} (0 - \alpha^{-n-\beta}) = 1$$

$$\Rightarrow \frac{c}{n+\beta} \alpha^{-n-\beta} = 1$$

$$\Rightarrow c = \frac{n+\beta}{\alpha^{-n-\beta}} = (n+\beta) \alpha^{n+\beta}$$

$$\text{Then, } K(\theta|y_n) = c \frac{1}{\theta^{n+\beta+1}} = \frac{(n+\beta) \alpha^{n+\beta}}{\theta^{n+\beta+1}}$$

The Bayes point estimate of θ such that the loss function is the squared error loss is

$$E(\theta|Y_n) = \int_{\alpha}^{\infty} \theta K(\theta|y_n) d\theta$$

$$= \int_{\alpha}^{\infty} \theta \frac{(n+\beta) \alpha^{n+\beta}}{\theta^{n+\beta+1}} d\theta$$

$$= (n+\beta) \alpha^{n+\beta} \int_{\alpha}^{\infty} \theta^{-n-\beta} d\theta$$

$$= (n+\beta) \alpha^{n+\beta} \left[\frac{\theta^{-n-\beta+1}}{-n-\beta+1} \right]_{\alpha}^{\infty}$$

We have proved in example (3.5) page:55

that $Y = \text{Maximum}(X_1, X_2, \dots, X_n)$ is a sufficient statistic

then

$$k(\theta|x) \propto f_{Y_n}(y_n|\theta) h(\theta) \quad \text{page: 112}$$

$$\begin{aligned}
&= \frac{n+\beta}{-n-\beta+1} \alpha^{n+\beta} (0 - \alpha^{-n-\beta+1}) \\
&= \frac{n+\beta}{-n-\beta+1} \alpha^{n+\beta} (-\alpha^{-n-\beta+1}) \\
&= \frac{n+\beta}{-n-\beta+1} (-\alpha^{n+\beta-n-\beta+1}) \\
&= \frac{n+\beta}{-n-\beta+1} (-\alpha) = \frac{n+\beta}{(n+\beta-1)} (-\alpha) = \frac{n+\beta}{(n+\beta-1)} \alpha \quad \dots (1)
\end{aligned}$$

5.5: In Exercise 5.4, let $n=4$ from the uniform pdf $f(x, \theta) = \frac{1}{\theta}, 0 < x < \theta$, and the prior θ pdf be $g(\theta) = \frac{2}{\theta^3}, 1 < \theta < \infty$, zero elsewhere. Find:

- (a) The Bayesian estimator $\delta(Y_4)$ of θ , based upon the sufficient statistic Y_4 , using the loss function $[\theta - \delta(Y_4)]^2$.
- (b) The Bayesian estimator $\delta(Y_4)$ of θ , based upon the sufficient statistic Y_4 , using the loss function $|\delta(Y_4) - \theta|$.

in 5.4 if $n=4$ $f(x) = \frac{1}{\theta}, 0 < x < \theta$

$$g(\theta) = \frac{2}{\theta^3}, 1 < \theta < \infty \Rightarrow g(\theta) = \frac{\beta \alpha^\beta}{\theta^{\beta+1}} = \frac{2(1)^2}{\theta^{2+1}} = \frac{2}{\theta^3}$$

(a) Y_4 is the maximum order statistic in the sample X_1, X_2, X_3, X_4

To find the Bayesian estimate of θ , we substitute in (1) by $n=4, \beta = 2$ and $\alpha = 1$

$$E(\theta|Y_4) = \frac{n+\beta}{(n+\beta-1)} \alpha^{\frac{4+2}{(4+2-1)}} (1) = \frac{6}{5}$$

(b) The Bayesian estimator (Y_4) of θ , based upon the sufficient statistic Y_4 , using the loss function $|\delta(y_4) - \theta|$.

it is the median m of $K(\theta|y_n)$ $\alpha < \theta < \infty$ which is the solution of

$$\begin{aligned}
&\int_{\alpha}^m K(\theta|y_n) d\theta = \frac{1}{2} \\
&\Rightarrow \int_{\alpha}^m \frac{(n+\beta) \alpha^{n+\beta}}{\theta^{n+\beta+1}} d\theta = \frac{1}{2} \quad (n=4, \beta = 2 \text{ and } \alpha = 1) \\
&\Rightarrow \int_1^m \frac{6}{\theta^7} d\theta = \frac{1}{2}
\end{aligned}$$

$$\Rightarrow \int_1^m 6\theta^{-7} d\theta = \frac{1}{2}$$

$$\Rightarrow \left. \frac{6\theta^{-6}}{-6} \right|_1^m = \frac{1}{2}$$

$$\Rightarrow -(m^{-6} - 1) = \frac{1}{2} \Rightarrow 1 - m^{-6} = \frac{1}{2} \Rightarrow m^{-6} = \frac{1}{2} \Rightarrow m^6 = 2 \Rightarrow m = 1.1225$$

5.6: $f(x|\theta) = \theta e^{-\theta x}$

$$h(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta\beta}, \theta > 0$$

(a) The posterior distribution of θ

$$L(x|\theta) = \prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^n \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^n x_i}$$

$$K(\theta|X) \propto \theta^n e^{-\theta \sum_{i=1}^n x_i} \theta^{\alpha-1} e^{-\theta\beta} = \theta^{n+\alpha-1} e^{-\theta(\beta + \sum_{i=1}^n x_i)}$$

$$\text{Then } \theta|X \sim \text{Gamma} \left(n + \alpha, \frac{1}{\beta + \sum_{i=1}^n x_i} \right)$$

(b) The Bayes Point estimate of θ use $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$

$$E(\theta|X) = \frac{n+\alpha}{\beta + \sum_{i=1}^n x_i}$$

(c) If $X_1=2.5$ $X_2=3.61$ $X_3=4.8$ $X_4=2.74$ $X_5=3.95$ and $\alpha=2$ $\beta=4$. Calculate (b).

$$\sum_{i=1}^5 x_i = 17.6, n=5, \alpha=2 \text{ and } \beta=4$$

$$E(\theta|X) = \frac{n+\alpha}{\beta + \sum_{i=1}^n x_i} = \frac{5+2}{17.6+4} = 0.3241$$