Ch.4: Motion in Two Dimensions

Physics 103: Classical Mechanics

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2025

Outline

1. The Position, Velocity, and Acceleration Vectors

2. Two-Dimensional Motion with Constant Acceleration

3. Projectile Motion

4. Uniform Circular Motion

5. Suggested Problems

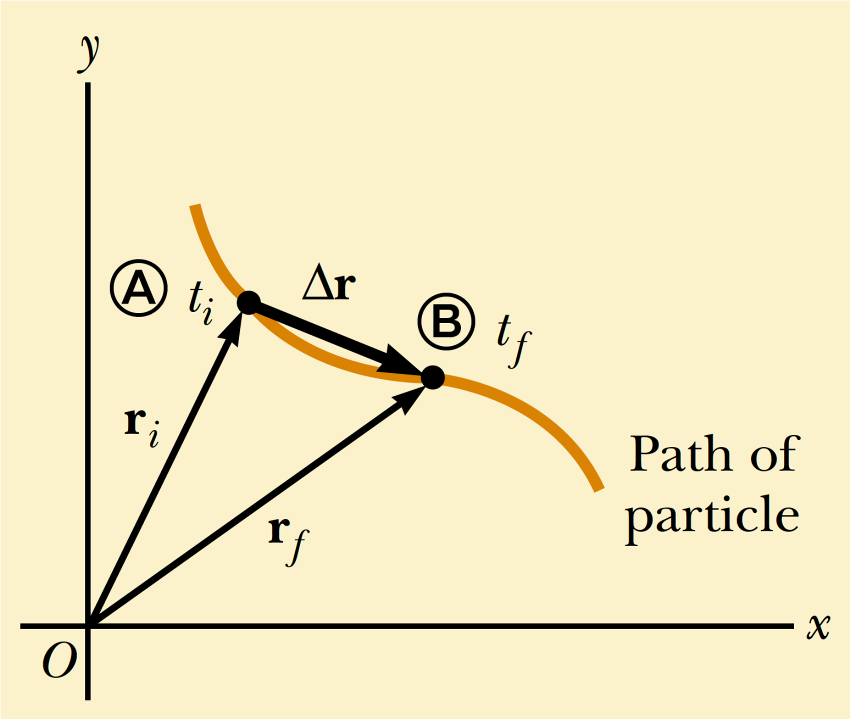
1.1 Position and Displacement Vectors

- The **position** vector $\vec{\mathbf{r}}$ of a particle describes its location in space relative to a chosen origin,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

- The **displacement** vector $\Delta\vec{r}$ of a particle is the change in its position:

$$\begin{aligned}\Delta\vec{r} &= \vec{r}_f - \vec{r}_i \\ &= (x_f - x_i)\hat{\imath} + (y_f - y_i)\hat{\jmath} \\ &= \Delta x\hat{\imath} + \Delta y\hat{\jmath}\end{aligned}$$



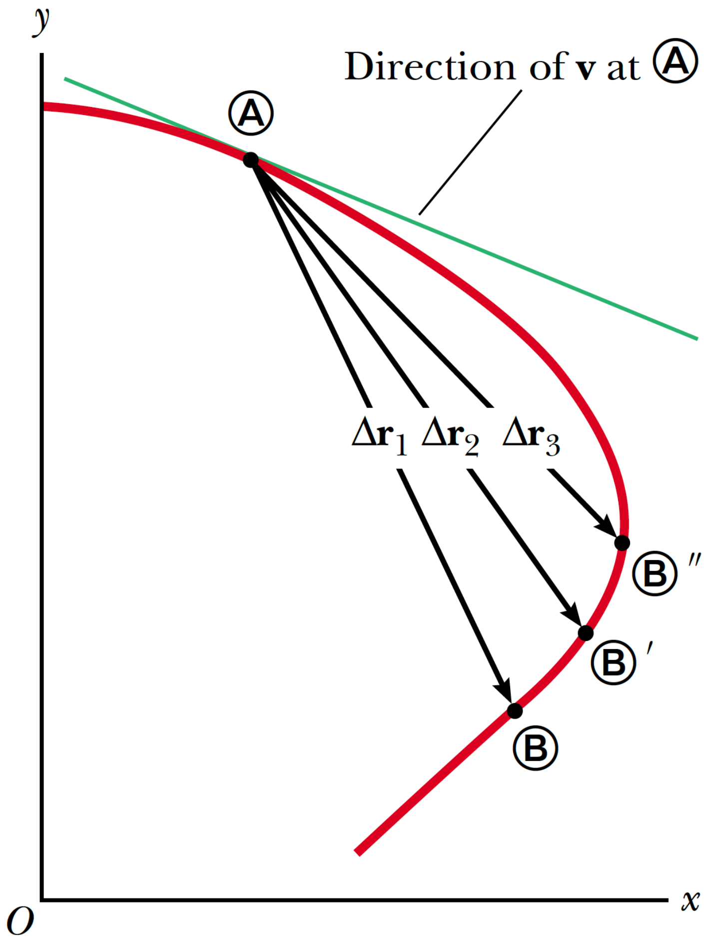
1.2 Velocity

- **Average** velocity is defined as the change in position divided by the time interval over which the change occurs:

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

- **Instantaneous** velocity is the limit of the average velocity as the time interval approaches zero:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



1.3 Acceleration

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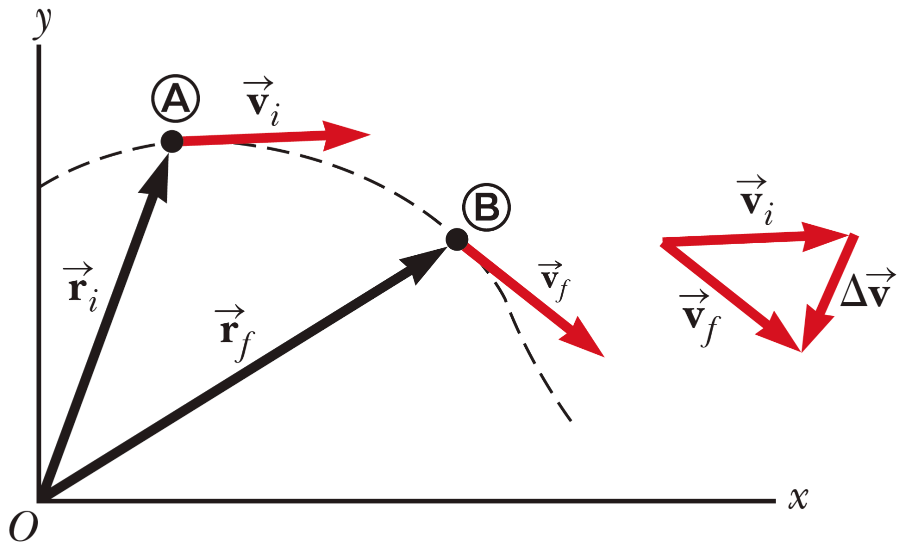
- **Average** acceleration is defined as the change in velocity divided by the time interval over which the change occurs:

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

- **Instantaneous** acceleration is the limit of the average acceleration as the time interval approaches zero:

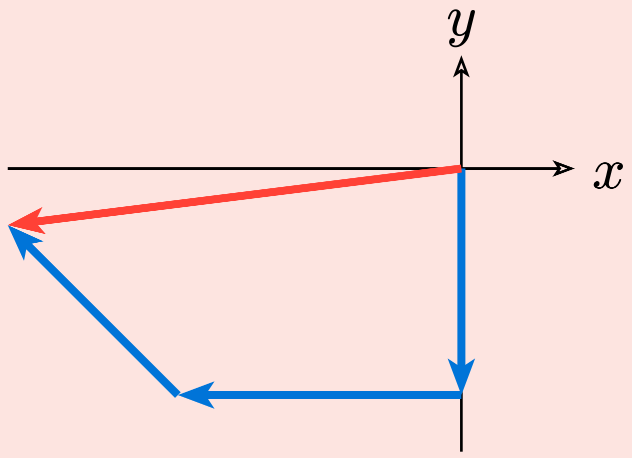
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$



1.4 Example

Problem 1.1

A motorist drives south at 20 m/s for 3 min, then turns west and travels at 25 m/s for 2 min, and finally travels northwest at 30 m/s for 1 min. For this 6 min trip, find:
(a) the total vector displacement,
(b) the average speed, and
(c) the average velocity.
Let the positive x axis point east.



1.4 Example

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Answer 1.1

(a) The total vector displacement can be found by breaking each segment of the trip into its vector components and then adding them up.

- Southward displacement:

$$\vec{d}_1 = 0\hat{\imath} + (-20 \text{ m/s} \times 180 \text{ s})\hat{\jmath} = -3600 \text{ m } \hat{\jmath}$$

- Westward displacement:

$$\vec{d}_2 = (-25 \text{ m/s} \times 120 \text{ s})\hat{\imath} + 0\hat{\jmath} = -3000 \text{ m } \hat{\imath}$$

- Northwestward displacement ($\theta = 180° - 45° = 135°$):

$$\vec{d}_3 = [30\cos 135° \times 60] \text{ m } \hat{\imath} + [30\sin 135° \times 60] \text{ m } \hat{\jmath}$$
$$= [-1273\hat{\imath} + 1273\hat{\jmath}] \text{ m}$$

1.4 Example

- Total displacement:

$$\vec{d} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3$$
$$= (0 - 3000 - 1273)\hat{\imath} + (-3600 + 0 + 1273)\hat{\jmath} = (-4273\hat{\imath} - 2327\hat{\jmath})\ \text{m}$$

- The magnitude of the total displacement is given by:

$$\left|\vec{d}\right| = \sqrt{(-4273)^2 + (-2327)^2} = 4906\ \text{m}$$

- The direction of the displacement vector is given by:

$$\theta = \tan^{-1}\left(\frac{-2327}{-4273}\right) = 208.7^\circ \qquad (\text{Third Quadrant: } 28.7^\circ + 180^\circ)$$

1.4 Example

(b) The average speed is the total distance traveled divided by the total time.

$$\begin{aligned} \text{Total distance} &= 3600 + 3000 + \sqrt{1273^2 + 1273^2} = 8400 \quad \text{m} \\ \text{Average speed} &= (8400 \text{ m})/(360 \text{ s}) = 23.3 \text{ m/s} \end{aligned}$$

(c) The average velocity is displacement divided by the total time.

$$\bar{\vec{v}} = \frac{\vec{d}}{\Delta t} = \frac{-4273\hat{\imath} - 2327\hat{\jmath}}{360} = (-11.9\hat{\imath} - 6.5\hat{\jmath}) \text{ m/s}$$

1.4 Example

Example 1.1

A particle moves in the xy plane with a position vector given by

$$\vec{r}(t) = (3t^2 - 4t)\hat{\imath} + (3t - 2)\hat{\jmath},$$

where $\vec{r}$ is in meters and t is in seconds.

a) Find the Instantaneous velocity at $t = 3$ s, including its magnitude and direction.

b) Find the Instantaneous acceleration at $t = 3$ s, including its magnitude and direction.

1.4 Example

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Solution 1.1

(a)

- Instantaneous velocity is given by:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[(3t^2 - 4t)\hat{\imath} + (3t - 2)\hat{\jmath}] = (6t - 4)\hat{\imath} + 3\hat{\jmath}$$

- At $t = 3s$,

$$\vec{v}(3) = (6(3) - 4)\hat{\imath} + 3\hat{\jmath} = 14\hat{\imath} + 3\hat{\jmath}$$

- The magnitude of the instantaneous velocity is given by:

$$|\vec{v}(3)| = \sqrt{(14)^2 + (3)^2} = \sqrt{196 + 9} = \sqrt{205} = 14.3 \;\; \text{m/s}$$

- The direction of the instantaneous velocity is given by:

$$\theta = \tan^{-1}\left(\frac{3}{14}\right) = 12.2^\circ$$

1.4 Example

(b)

- Instantaneous acceleration is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[(6t - 4)\hat{\imath} + 3\hat{\jmath}] = 6\hat{\imath}$$

- We see that the acceleration is always constant at all times.
- The magnitude of the instantaneous acceleration is given by:

$$|\vec{a}| = 6 \text{ m/s}^2$$

- The direction of the instantaneous acceleration is given by:

$$\theta = 0°$$

- Notice that the direction of acceleration is different from the direction of velocity at $t = 3s$.

1. The Position, Velocity, and Acceleration Vectors

2. Two-Dimensional Motion with Constant Acceleration

3. Projectile Motion

4. Uniform Circular Motion

5. Suggested Problems

2.1 Remember

Equations of Motion at Constant Acceleration in One Dimension	
$v_{xf} = v_{xi} + a_x t$	(1)
$x_f = x_i + \frac{1}{2}\left(v_{xf} + v_{xi}\right)t$	(2)
$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$	(3)
$v_{xf}^2 = v_{xi}^2 + 2a_x\left(x_f - x_i\right)$	(4)

2.2 In Two Dimensions

Equations of Motion at Constant Acceleration in Two Dimension
$\vec{\boldsymbol{v}}_f = \vec{\boldsymbol{v}}_i + \vec{\boldsymbol{a}}t \qquad (1)$
$\vec{\boldsymbol{r}}_f = \vec{\boldsymbol{r}}_i + \vec{\boldsymbol{v}}_i t + \frac{1}{2}\vec{\boldsymbol{a}}t^2 \qquad (3)$

Remember

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

$$\vec{\boldsymbol{v}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}$$

$$\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}}$$

Therefore:

$$\vec{\boldsymbol{v}}_f = (v_{xi} + a_x t)\hat{\boldsymbol{\imath}} + \left(v_{yi} + a_y t\right)\hat{\boldsymbol{\jmath}}$$

$$\vec{\boldsymbol{r}}_f = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\hat{\boldsymbol{\imath}} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\hat{\boldsymbol{\jmath}}$$

Example 2.2

A particle starts from the origin at $t = 0$ with an initial velocity having an x component of $20 \quad \text{m/s}$ and a y component of $-15 \quad \text{m/s}$. The particle moves in the xy plane with an x component of acceleration only, given by $a_x = 4 \quad \text{m/s}^2$.

(A) Determine the components of the velocity vector at any time and the total velocity vector at any time.

2.3 Examples

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Solution 2.2

X	x_i	x_f	v_{xi}	v_{xf}	a_x	t_i	t_f
Value	0	-	20 m/s	?	4 m/s^2	0	t
Y	y_i	y_f	v_{yi}	v_{yf}	a_y		
Value	0	-	-15 m/s	?	0		

- The components of the velocity vector at any time (t) are given by:

$$\text{X:} \quad v_{xf} = v_{xi} + a_x t = (20 + 4t) \quad \text{m/s}$$

$$\text{Y:} \quad v_{yf} = v_{yi} + a_y t = -15 \quad \text{m/s}$$

- Therefore,

$$\vec{\boldsymbol{v}}_f = v_{xf}\hat{\boldsymbol{\imath}} + v_{yf}\hat{\boldsymbol{\jmath}} = [(20 + 4t)\hat{\boldsymbol{\imath}} - 15\hat{\boldsymbol{\jmath}}] \quad \text{m/s}$$

2.3 Examples

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Example 2.2

(B) Calculate the velocity and speed of the particle at $t = 5s$.

2.3 Examples

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Solution 2.2

- At $t = 5 \;\; \text{s}$,

$$v_{xf} = 20 + 4(5) = 40 \;\; \text{m/s}$$
$$v_{yf} = -15 \;\; \text{m/s}$$

- Therefore,

$$\vec{\boldsymbol{v}}_f = (40\hat{\boldsymbol{\imath}} - 15\hat{\boldsymbol{\jmath}}) \;\; \text{m/s}$$

$$\text{Speed} = |\vec{\boldsymbol{v}}_f| = \sqrt{(40 \;\; \text{m/s})^2 + (-15 \;\; \text{m/s})^2} = 42.7 \;\; \text{m/s}$$

- The direction of the velocity vector is given by

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(-\frac{15}{40}\right) = -21^\circ.$$

2.3 Examples

Example 2.2

(C) Determine the x and y coordinates of the particle at any time t and the position vector at this time.

Solution 2.2

- From the equations of motion,

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 = 0 + 20t + 2t^2 = \left(20t + 2t^2\right)\ \text{m}$$

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2 = 0 - 15t + 0 = \left(-15t\right)\ \text{m}$$

- The position vector at any time t is given by

$$\vec{\boldsymbol{r}}_f = x_f\hat{\boldsymbol{\imath}} + y_f\hat{\boldsymbol{\jmath}} = \left[\left(20t + 2t^2\right)\hat{\boldsymbol{\imath}} - 15t\hat{\boldsymbol{\jmath}}\right]\ \text{m}$$

- Therefore, at $t = 5s$,

$$\vec{\boldsymbol{r}}_f = \left[\left(20(5) + 2(5)^2\right)\hat{\boldsymbol{\imath}} - 15(5)\hat{\boldsymbol{\jmath}}\right]\ \text{m} = \left(150\hat{\boldsymbol{\imath}} - 75\hat{\boldsymbol{\jmath}}\right)\ \text{m}$$

Problem 2.2

At $t = 0$, a particle moving in the xy plane with constant acceleration has a velocity of $\vec{v}_i = (3\hat{\imath} - 2\hat{\jmath})$ m/s and is at the origin. At $t = 3$ s, the particle's velocity is $\vec{v} = (9\hat{\imath} + 7\hat{\jmath})$ m/s. Find:

(a) the acceleration of the particle and

(b) its coordinates at any time t.

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Answer 2.2

(a) From Eq(1),

$$\vec{\boldsymbol{a}} = \frac{\vec{\boldsymbol{v}}_f - \vec{\boldsymbol{v}}_i}{t} = \frac{(9-3)\hat{\boldsymbol{\imath}} + (7-(-2))\hat{\boldsymbol{\jmath}}}{3} = (2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\ \mathrm{m/s^2}$$

(b) From Eq(3),

$$\vec{\boldsymbol{r}}_f = \vec{\boldsymbol{r}}_i + \vec{\boldsymbol{v}}_i t + \frac{1}{2}\vec{\boldsymbol{a}}t^2$$

Since the particle is at the origin at $t = 0$, we have $\vec{\boldsymbol{r}}_i = 0$. Therefore,

$$\vec{\boldsymbol{r}}_f = \vec{\boldsymbol{v}}_i t + \frac{1}{2}\vec{\boldsymbol{a}}t^2$$

Substituting the known values,

$$\vec{r}_f = (3\hat{\imath} - 2\hat{\jmath})t + \frac{1}{2}(2\hat{\imath} + 3\hat{\jmath})t^2$$

Therefore,

$$x = 3t + t^2, \quad y = -2t + \left(\frac{3}{2}\right)t^2$$

2.3 Examples

Problem 2.3

The vector position of a particle varies in time according to the expression

$$\vec{\boldsymbol{r}} = \left(3\hat{\boldsymbol{\imath}} - 6t^2\hat{\boldsymbol{\jmath}}\right)\ \mathrm{m}.$$

(a) Find expressions for the velocity and acceleration as functions of time.

(b) Determine the particle's position and velocity at $t = 1s$.

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Answer 2.3

(a) The velocity is given by the time derivative of the position vector:

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = (0\hat{\boldsymbol{\imath}} - 12t\hat{\boldsymbol{\jmath}})\ \mathrm{m/s}$$

The acceleration is given by the time derivative of the velocity vector:

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = (0\hat{\boldsymbol{\imath}} - 12\hat{\boldsymbol{\jmath}})\ \mathrm{m/s^2}$$

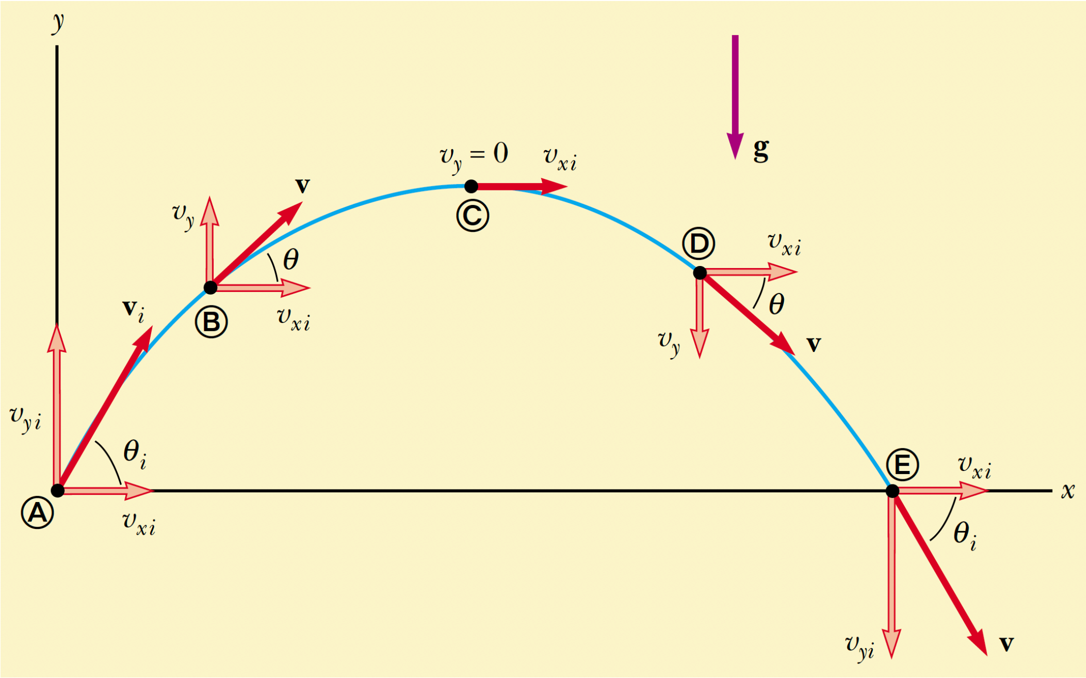
(b) At $t = 1$ s,

$$\vec{\boldsymbol{r}} = (3\hat{\boldsymbol{\imath}} - 6(1)^2\hat{\boldsymbol{\jmath}})\ \mathrm{m} = (3\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}})\ \mathrm{m}$$

$$\vec{\boldsymbol{v}} = (0\hat{\boldsymbol{\imath}} - 12(1)\hat{\boldsymbol{\jmath}})\ \mathrm{m/s} = -12\hat{\boldsymbol{\jmath}}\ \mathrm{m/s}$$

3.1 Assumptions

- The only force acting on the projectile is the force of *gravity*, which acts downward.
- The acceleration of the projectile is *constant* and equal to $\vec{a} = -g\hat{j}$.
- The horizontal component of the velocity remains *constant* throughout the flight because there is no horizontal acceleration (ignoring air resistance).



$$v_{xi} = v_i \cos\theta_i \qquad a_x = 0$$

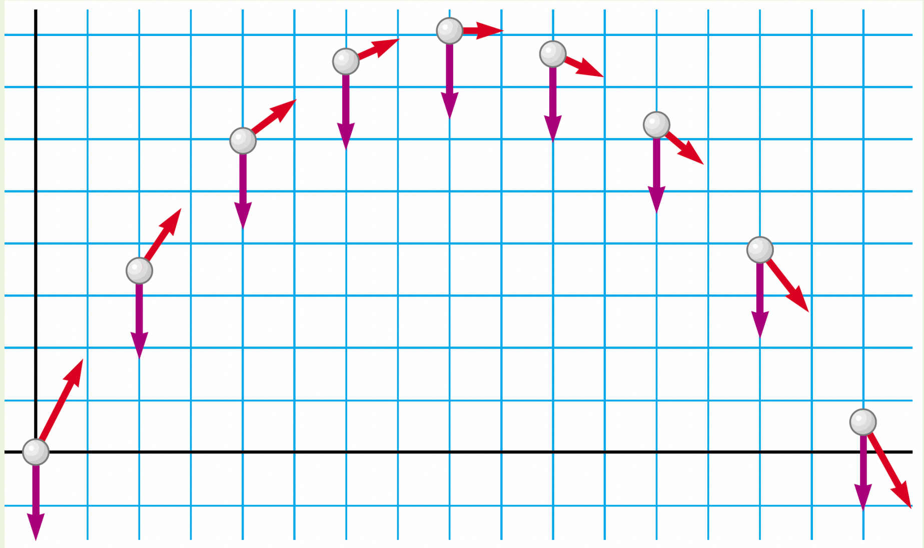
$$v_{yi} = v_i \sin\theta_i \qquad a_y = -g$$

3.2 Examples

Example 3.3

A ball is thrown in such a way that its initial vertical and horizontal components of velocity are 40 m/s and 20 m/s, respectively.

Estimate the total time of flight and the distance the ball is from its starting point when it lands.



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Solution 3.3

- The total time of flight can be estimated by calculating the time it takes for the ball to reach its maximum height and then doubling this time (the time to go up equals the time to come down).
- The time to reach maximum height can be found from the vertical component of motion. At maximum height, the vertical component of velocity is zero.
- Therefore, the speed after each second is estimated as follows ($g \approx 10$ m/s^2):

Time (s)	0	1	2	3	4
Vertical Velocity (m/s)	40	30	20	10	0

- From the table, we see that it takes about 4 s to reach maximum height. Therefore, the total time of flight is about $2 \times 4 = 8$ s.

3.2 Examples

- The horizontal distance traveled by the ball can be estimated from the horizontal component of motion.
- The horizontal component of velocity remains constant throughout the flight because there is no horizontal acceleration (ignoring air resistance).
- Therefore, the horizontal distance traveled by the ball is given by:

$$x = v_{xi}t = (20\ \text{m/s})(8s) = 160\ \text{m}$$

3.3 Equations of Motion for a Projectile

Using the assumptions:

$$v_{xi} = v_i \cos\theta_i \qquad a_x = 0$$
$$v_{yi} = v_i \sin\theta_i \qquad a_y = -g$$

we can summarize the equations of motion for a projectile as follows:

Horizontal Direction	Vertical Direction
$v_{xf} = v_{xi} = v_i \cos\theta_i$ (h1)	$v_{yf} = v_i \sin\theta_i - gt$ (v1)
$x_f = x_i + (v_i \cos\theta_i)t$ (h2)	$y_f = y_i + (v_i \sin\theta_i)t - \frac{1}{2}gt^2$ (v2)
	$v_{yf}^2 = v_i^2 \sin^2\theta_i - 2g\left(y_f - y_i\right)$ (v3)

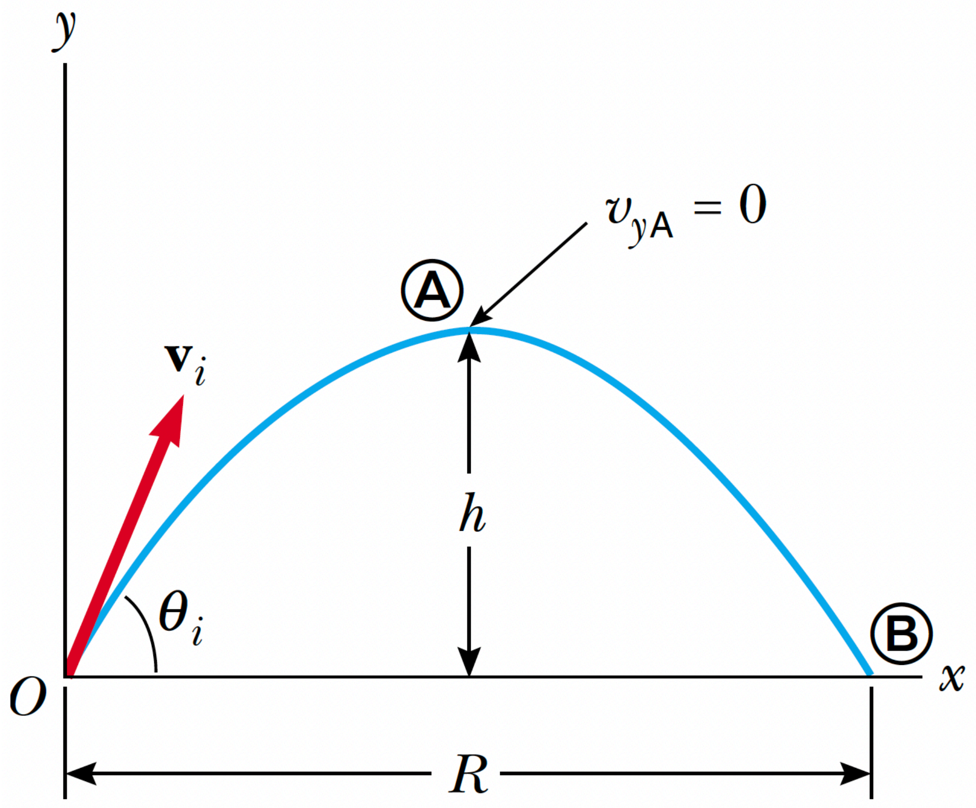
3.4 Horizontal Range and Maximum Height

- The **maximum height** h of a projectile is the highest vertical position it reaches during its flight.

$$h = y_{\text{max}} - y_i$$

- At maximum height, $v_{yf} = 0$, therefore, using Eq(v3) we get:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$



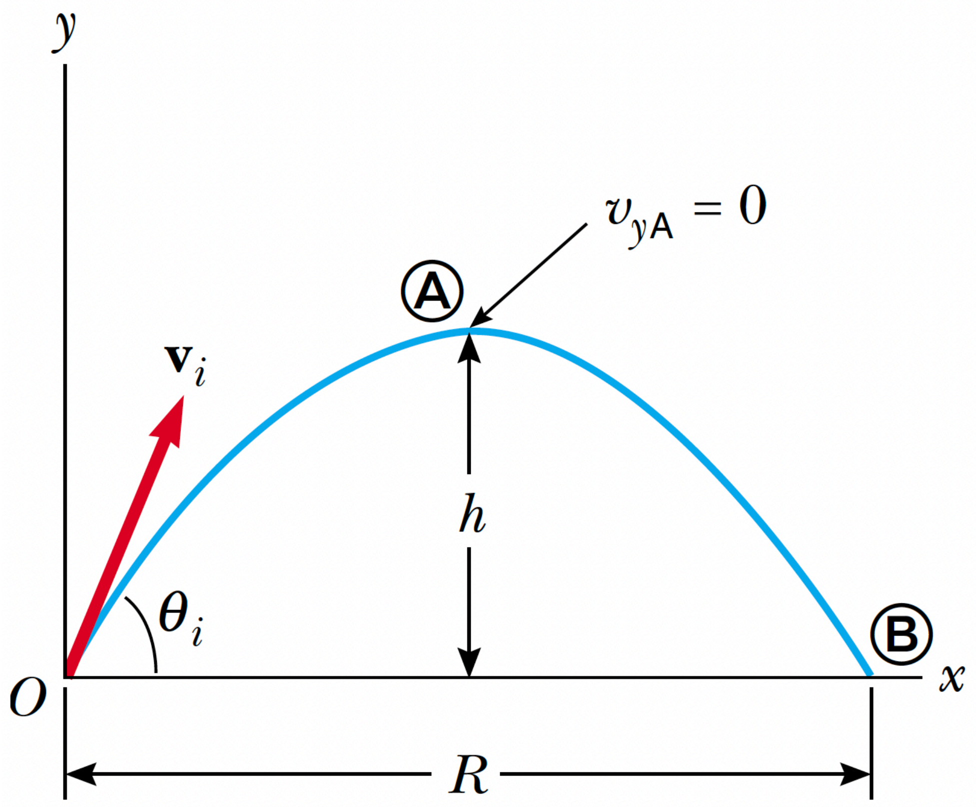
3.4 Horizontal Range and Maximum Height

- The **horizontal range** R of a projectile is the horizontal distance it travels during its time of flight.

$$R = x_f - x_i$$

- From Eq(h2) and Eq(v1), the horizontal range can be expressed as:

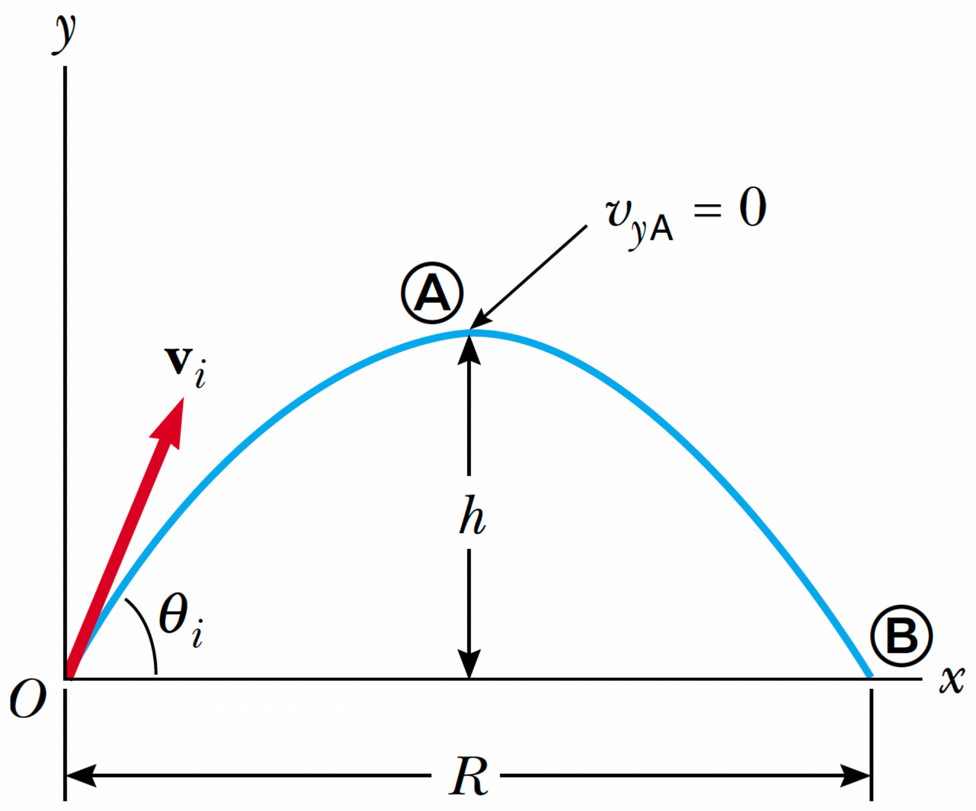
$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$



3.4 Horizontal Range and Maximum Height

- The **time of flight** T of a projectile launched from and landing at the same height is given by:

$$T = \frac{2v_i \sin\theta_i}{g}$$



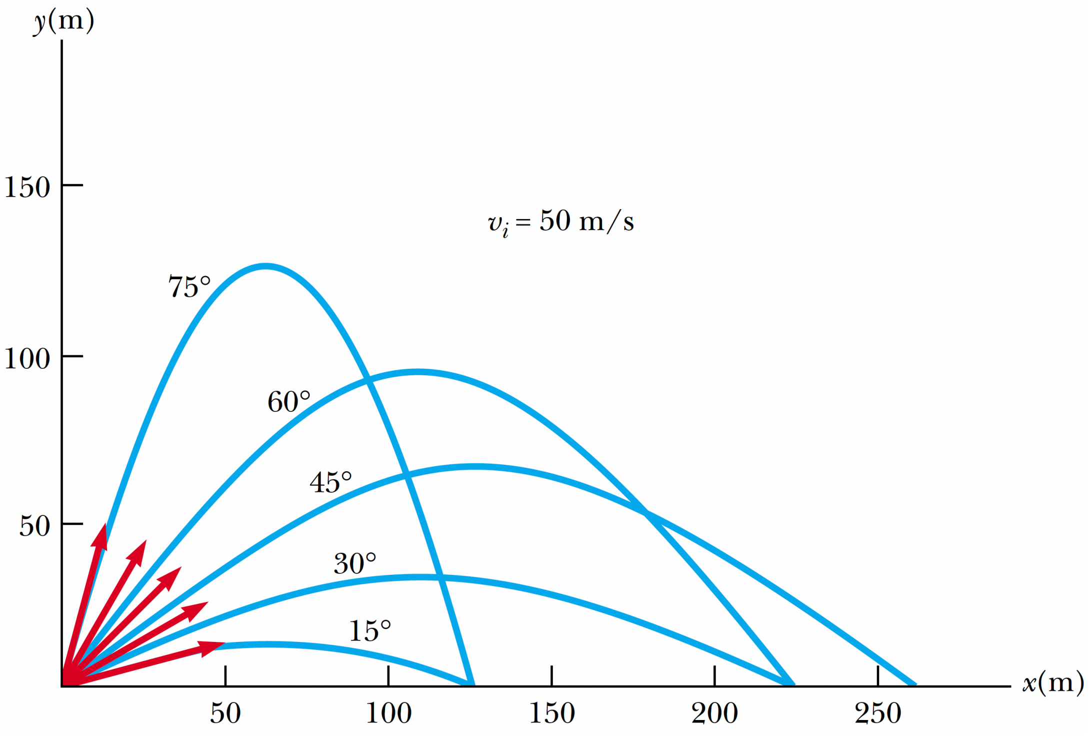
3.4 Horizontal Range and Maximum Height

- The angle θ_i that maximizes the maximum height h of a projectile is 90°.

$$h_{\max} = \frac{v_i^2}{2g}$$

- The angle θ_i that maximizes the horizontal range R of a projectile is 45°.

$$R_{\max} = \frac{v_i^2}{g}$$



3.4 Horizontal Range and Maximum Height

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Example 3.4

A long-jumper leaves the ground at an angle of 20° above the horizontal and at a speed of 11 m/s.

(A) How far does he jump in the horizontal direction?

(B) What is the maximum height reached?

3.4 Horizontal Range and Maximum Height

Solution 3.4

(A) The horizontal range R of the jump can be found from:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11 \text{ m/s})^2 \sin 40°}{9.8 \text{ m/s}^2} = 7.94 \text{ m}$$

(B) The maximum height H reached can be found from:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g} = \frac{(11 \text{ m/s})^2 \sin^2 20°}{2 \times 9.8 \text{ m/s}^2} = 0.72 \text{ m}$$

5. Suggested Problems

1, 3, 5, 6, 8, 14, 15, 17, 19, 20, 22, 23, 25, 29