Integral Calculus

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Main Contents

Definition

Let $\{a_1, a_2, \dots, a_n\}$ *be a set of numbers. The symbol* $\sum_{k=1}^{n} a_k$ *represents their sum:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n.$$

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Evaluate the sum.

1. $\sum_{i=1}^{3} i^3$
2. $\sum_{j=2}^{5} (j^2 + 1)$

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Solution:

1. $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36.$
2.
3. $\sum_{j=2}^{5} (j^2 + 1) = (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) = 58$

Section 1: Summation Notation

Theorem

Let $\{a_1, a_2, \ldots, a_n\}$ *and* $\{b_1, b_2, \ldots, b_n\}$ *be sets of real numbers. If* n *is any positive integer, then*

1. $\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{n\text{-times}} = nc$ *for any* $c \in \mathbb{R}$.
2. $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$.
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Evaluate the sum $\sum_{k=1}^{4} (k^2 + 2k + 3)$.

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Evaluate the sum $\sum_{k=1}^{4} (k^2 + 2k + 3)$.

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$$\sum_{k=1}^{4} (k^2 + 2k + 3) = \sum_{k=1}^{4} k^2 + 2\sum_{k=1}^{4} k + \sum_{k=1}^{4} 3$$

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Evaluate the sum $\sum_{k=3}^{10} 15$

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Example

Evaluate the sum $\displaystyle\sum_{k=3}^{10} 15$

Solution: $\displaystyle\sum_{k=1}^{10} 15 = \sum_{k=1}^{2} 15 + \sum_{k=3}^{10} 15 \Rightarrow 10 \times 15 = 2 \times 15 + \sum_{k=3}^{10} 15$

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Solution: $\sum_{k=1}^{10} 15 = \sum_{k=1}^{2} 15 + \sum_{k=3}^{10} 15 \Rightarrow 10 \times 15 = 2 \times 15 + \sum_{k=3}^{10} 15 \Rightarrow \sum_{k=3}^{10} 15 = 150 - 30 = 120$

Section 1: Summation Notation

Theorem

1. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

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Evaluate the sum.

1. $\sum_{k=1}^{100} k$
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1. $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050.$

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Evaluate the sum.

1. $\sum_{k=1}^{100} k$
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Solution:

1. $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050.$
2. $\sum_{k=1}^{10} k^2 = \frac{10(11)(21)}{6} = 385.$

Theorem

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Example

Evaluate the sum $\sum_{k=5}^{100} k$.

Theorem

1. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
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Evaluate the sum.

1. $\sum_{k=1}^{100} k$
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Example

Evaluate the sum $\sum_{k=5}^{100} k$.

Solution: $\sum_{k=1}^{100} k = \sum_{k=1}^{4} k + \sum_{k=5}^{100} k$

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Evaluate the sum.

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Example

Evaluate the sum $\sum_{k=5}^{100} k$.

Solution: $\sum_{k=1}^{100} k = \sum_{k=1}^{4} k + \sum_{k=5}^{100} k \Rightarrow \frac{100(100+1)}{2} = \frac{4(4+1)}{2} + \sum_{k=5}^{100} k$

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Evaluate the sum $\sum_{k=5}^{100} k.$

Solution: $\sum_{k=1}^{100} k = \sum_{k=1}^{4} k + \sum_{k=5}^{100} k \Rightarrow \frac{100(100+1)}{2} = \frac{4(4+1)}{2} + \sum_{k=5}^{100} k \Rightarrow 5050 = 10 + \sum_{k=5}^{100} k$

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1. $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050.$
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Example

Evaluate the sum $\sum_{k=5}^{100} k.$

Solution: $\sum_{k=1}^{100} k = \sum_{k=1}^{4} k + \sum_{k=5}^{100} k \Rightarrow \frac{100(100+1)}{2} = \frac{4(4+1)}{2} + \sum_{k=5}^{100} k \Rightarrow 5050 = 10 + \sum_{k=5}^{100} k \Rightarrow \sum_{k=5}^{100} k = 5050 - 10 = 5040$

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Express the sum $\sum_{k=1}^{n}(k^2 - k - 1)$ *in terms of* n

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Express the sum $\sum_{k=1}^{n}(k^2 - k - 1)$ *in terms of* n

Solution:

$$\sum_{k=1}^{n}(k^2 - k - 1) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

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Express the sum $\sum_{k=1}^{n}(k^2-k-1)$ *in terms of* n

Solution:

$$\sum_{k=1}^{n}(k^2-k-1)=\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k-\sum_{k=1}^{n}1=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}-n$$

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Express the sum $\sum_{k=1}^{n}(k^2 - k - 1)$ *in terms of* n

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Example

Choose the correct answer.

- *If* $\sum_{k=1}^{n}(k+\alpha) = \frac{n^2}{2} (n \geq 1)$, *then the value of* α *is equal to*

(a) $-\frac{n}{2}$ *(b)* $\frac{1}{2}$ *(c)* $-\frac{1}{2}$ *(d)* 1

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Solution:

$$\sum_{k=1}^{n}(k+\alpha) = \frac{n^2}{2} \Rightarrow \sum_{k=1}^{n} k + \sum_{k=1}^{n} \alpha = \frac{n^2}{2}$$

$$\Rightarrow \frac{n(n+1)}{2} + \alpha n = \frac{n^2}{2}$$

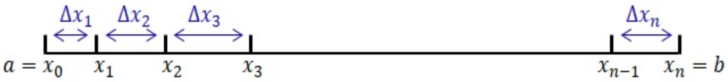
$$\Rightarrow \frac{n+1}{2} + \alpha = \frac{n}{2} \qquad \text{divide all terms by} n$$

$$\Rightarrow \alpha = \frac{n}{2} - \frac{n+1}{2} = \frac{n-n-1}{2} \Rightarrow \alpha = -\frac{1}{2}$$

Definition

A partition P of the closed interval $[a, b]$ is a finite set of points $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\ .$$



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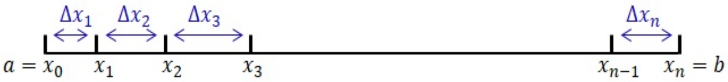

Notes:

1. The division of the interval $[a, b]$ by a partition P generates n subintervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$.
2. The length of the subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
3. The union of all subintervals gives the whole interval $[a, b]$.

Definition

The norm of a partition P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$ *i.e.,*

$$\| P \| = max\{\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n\}.$$

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Example

If $P = \{0, 1.2, 2.3, 3.6, 4\}$ *is a partition of the interval* $[0, 4]$*, find the norm of* P*.*

Section 2: Riemann Sum and Area

Definition

The norm of a partition P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$ i.e.,

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Example

If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0, 4]$, find the norm of P.

Solution:
We need to find the subintervals and their lengths.

Subinterval $[x_{k-1}, x_k]$	Length Δx_k
$[0, 1.2]$	$1.2 - 0 = 1.2$
$[1.2, 2.3]$	$2.3 - 1.2 = 1.1$
$[2.3, 3.6]$	$3.6 - 2.3 = 1.3$
$[3.6, 4]$	$4 - 3.6 = 0.4$

The norm of P is the largest length among

$$\| P \| = max\{1.2, 1.1, 1.3, 0.4\}.$$

Hence, $\| P \| = \Delta x_3 = 1.3$

Remark

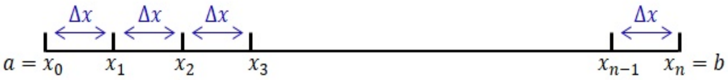
1. *The partition P of a closed interval* $[a, b]$ *is regular if* $\Delta x_0 = \Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = \Delta x$.
2. *For any positive integer n, if the partition P is regular then*

$$\Delta x = \frac{b-a}{n} \quad \textit{and} \quad x_k = x_0 + k\,\Delta x.$$

Let P be a regular partition of the interval $[a, b]$. Since $x_0 = a$ and $x_n = b$, then

$$\begin{aligned} x_1 &= x_0 + \Delta x\ , \\ x_2 &= x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x\ , \\ x_3 &= x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x. \end{aligned}$$

By continuing doing so, we have $x_k = x_0 + k\,\Delta x$.



Example

Define a regular partition P that divides the interval $[1, 4]$ into 4 subintervals.

Example

Define a regular partition P that divides the interval $[1,4]$ into 4 subintervals.

Solution: Since P is a regular partition of $[1,4]$ where $n=4$, then

$$\Delta x = \frac{4-1}{4} = \frac{3}{4} \text{ and } x_k = 1 + k\,\frac{3}{4}.$$

Therefore,

$$x_0 = 1$$

$$x_1 = 1 + 1\,\tfrac{3}{4} = \tfrac{7}{4}$$

$$x_2 = 1 + 2\left(\tfrac{3}{4}\right) = \tfrac{5}{2}$$

$$x_3 = 1 + 3\left(\tfrac{3}{4}\right) = \tfrac{13}{4}$$

$$x_4 = 1 + 4\left(\tfrac{3}{4}\right) = 4$$

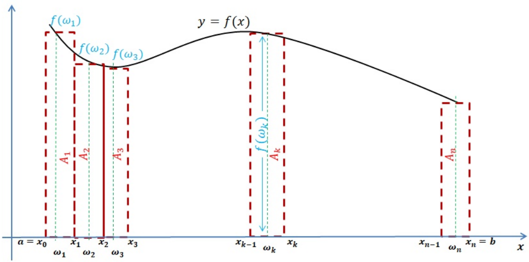
The regular partition is $P = \{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4\}$.

Section 2: Riemann Sum and Area

Definition

Let f be a bounded and defined function on a closed bounded interval $[a, b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a mark on the partition P where $\omega_k \in [x_{k-1}, x_k]$, $k = 1, 2, 3, \ldots, n$. Then the Riemann sum of f with respect to the partition P and the mark ω is

$$R(f, P, \omega) = \sum_{k=1}^{n} f(\omega_k)\Delta x_k.$$



$$
\begin{aligned}
A_1 &= f(\omega_1)\Delta x_1\\
A_2 &= f(\omega_2)\Delta x_2\\
A_3 &= f(\omega_3)\Delta x_3\\
&\vdots\\
A_n &= f(\omega_n)\Delta x_n\\
\Rightarrow A &= \sum_{k=1}^{n} A_k\\
\Rightarrow A &= \sum_{k=1}^{n} f(\omega_k)\Delta x_k
\end{aligned}
$$

$$A = \lim_{\|P\|\to 0} R(f, P, \omega) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\omega_k)\Delta x_k \qquad (1)$$

Example

Find a Riemann sum of the function $f(x) = 2x - 1$ *for the partition* $P = \{-2, 0, 1, 4, 6\}$ *of the interval* $[-2, 6]$ *by choosing the mark,*

1. *the left-hand endpoint,*
2. *the right-hand endpoint,*
3. *the midpoint.*

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1. *the left-hand endpoint,*
2. *the right-hand endpoint,*
3. *the midpoint.*

Solution:
(1) Choose the left-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	-2	$f(-2) = 2(-2) - 1 = -5$	-10
$[0, 1]$	$1 - 0 = 1$	0	$f(0) = 2(0) - 1 = -1$	-1
$[1, 4]$	$4 - 1 = 3$	1	1	3
$[4, 6]$	$6 - 4 = 2$	4	7	14
$R(f, P, \omega) = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				6

(2) Choose the right-hand endpoint of each subinterval. *Remember* $f(x) = 2x - 1$.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	0	$f(0) = 2(0) - 1 = -1$	-2
$[0, 1]$	$1 - 0 = 1$	1	1	1
$[1, 4]$	$4 - 1 = 3$	4	7	21
$[4, 6]$	$6 - 4 = 2$	6	11	22
$R(f, P, \omega) = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				42

[1] The midpoint of the subinterval $[x_{k-1}, x_k]$ is $\omega_k = \frac{x_{k-1}+x_k}{2}$.

(2) Choose the right-hand endpoint of each subinterval. *Remember* $f(x) = 2x - 1$.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\,\Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	0	$f(0) = 2(0) - 1 = -1$	-2
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$[1, 4]$	$4 - 1 = 3$	4	7	21
$[4, 6]$	$6 - 4 = 2$	6	11	22
$R(f, P, \omega) = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				42

(3) Choose the midpoint of each subinterval.[1]

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\,\Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	$\frac{-2+0}{2} = \frac{-2}{2} = -1$	$f(-1) = 2(-1) - 1 = -3$	-6
$[0, 1]$	$1 - 0 = 1$	$\frac{0+1}{2} = \frac{1}{2} = 0.5$	$f(0.5) = 2(0.5) - 1 = 0$	0
$[1, 4]$	$4 - 1 = 3$	$\frac{1+4}{2} = \frac{5}{2} = 2.5$	4	12
$[4, 6]$	$6 - 4 = 2$	$\frac{4+6}{2} = \frac{10}{2} = 5$	9	18
$R(f, P, \omega) = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				24

[1] The midpoint of the subinterval $[x_{k-1}, x_k]$ is $\omega_k = \frac{x_{k-1}+x_k}{2}$.

Example

Let A be the area under the graph of $f(x) = x + 1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right-hand endpoint of each subinterval.

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Solution: Since the partition P is regular, then

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \text{ and } x_k = x_0 + k\,\Delta x, \quad \text{where } x_0 = 1$$

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The mark ω is the right endpoint of each subinterval i.e., $\omega_k \in [x_{k-1}, x_k]$, so $\omega_k = x_k = 1 + \frac{2k}{n}$. From the given function, we have

$$f(\omega_k) = \omega_k + 1 = (1 + \frac{2k}{n}) + 1 = \frac{2k}{n} + 2\,.$$

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$$f(\omega_k) = \omega_k + 1 = (1 + \frac{2k}{n}) + 1 = \frac{2k}{n} + 2\,.$$

From the definition,

$$\begin{aligned} R(f,P,\omega) = \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \sum_{k=1}^{n} (\frac{2k}{n} + 2)\,\frac{2}{n} &= \frac{2}{n}\Big[\frac{2}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 2\Big] \\ &= \frac{2}{n}\Big[\frac{2}{n}\,\frac{n(n+1)}{2} + 2n\Big] \\ &= \frac{2}{n}\Big[(n+1) + 2n\Big] \\ &= \frac{2(n+1)}{n} + 4. \end{aligned}$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

$\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{\text{n-times}} = nc$ for any $c \in \mathbb{R}$.

Example

Let A be the area under the graph of $f(x) = x + 1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right-hand endpoint of each subinterval.

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From the definition,

$$R(f,P,\omega) = \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \sum_{k=1}^{n} (\frac{2k}{n} + 2)\frac{2}{n} = \frac{2}{n}\Big[\frac{2}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 2\Big]$$

$$= \frac{2}{n}\Big[\frac{2}{n}\,\frac{n(n+1)}{2} + 2n\Big]$$

$$= \frac{2}{n}\Big[(n+1) + 2n\Big]$$

$$= \frac{2(n+1)}{n} + 4.$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k \text{ for any } c \in \mathbb{R}.$$

$$\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{\text{n-times}} = nc \text{ for any } c \in \mathbb{R}.$$

Hence, $\lim_{n\to\infty} R(f,P,\omega) = \lim_{n\to\infty} \left(\frac{2(n+1)}{n} + 4\right) = \lim_{n\to\infty} \left(\frac{2(\frac{n}{n} + \frac{1}{n})}{\frac{n}{n}} + 4\right) = 2 + 4 = 6.$

Section 2: Riemann Sum and Area

Definition

For any function f bounded and defined on a closed bounded interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_k f(\omega_k)\Delta x_k, \ (\| P \| \to 0)$$

if the limit exists. The numbers a and b are called the limits of the integration.

Notes: The limit is over all pointed partitions $P = \{([x_{k-1}, x_k], \omega_k)\}_{1\leq k\leq n}$. When the limit exists, we say that f is Riemann integrable (or integrable) on $[a, b]$.

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By using Riemann sum evaluate the integral $\int_2^4 (x+2)\ dx$.

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Example

By using Riemann sum evaluate the integral $\int_2^4 (x+2)\ dx$.

Solution: Let $P = \{x_0, x_1, \dots, x_n\}$ be a regular partition of the interval $[2, 4]$, such that $\Delta x = \frac{4-2}{n} = \frac{2}{n}$ and $x_k = x_0 + k\Delta x$. Let the mark ω be the right endpoint of each subinterval, so $\omega_k = x_k = 2 + \frac{2k}{n}$ and then $f(\omega_k) = 2 + \frac{2k}{n} + 2 = 4 + \frac{2k}{n}$.
The Riemann sum of f for P is

$$R(f, P, \omega) = \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \sum_{k=1}^{n}\left(4 + \frac{2k}{n}\right)\frac{2}{n} = \frac{2}{n}\left(\sum_{k=1}^{n} 4 + \sum_{k=1}^{n}\frac{2k}{n}\right) = \frac{2}{n}\left(4n + \frac{2}{n}\frac{n(n+1)}{2}\right) = 8 + \frac{2(n+1)}{n}$$

From the definition,

$$\int_2^4 (x+2)\ dx = \lim_{n\to\infty}\left(8 + \frac{2(n+1)}{n}\right) = 8 + 2 = 10$$

Section 3: Properties of Definite Integrals

Theorem

(1) $\int_a^b c\ dx = c(b-a)$.

(2) If f and g are integrable on $[a, b]$, then $f+g$ and $f-g$ are integrable on $[a, b]$ and

$$\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x) \pm \int_a^b g(x)\ dx.$$

(3) If f is integrable on $[a, b]$ and $k \in \mathbb{R}$, then $k\ f$ is integrable on $[a, b]$ and

$$\int_a^b k\ f(x)\ dx = k\ \int_a^b f(x)\ dx.$$

(4) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

(5) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b\ f(x)\ dx \geq 0.$$

Theorem

(6) If f is integrable on the intervals $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

(7) If $a < b$ we will denote $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$.

(8) If $f(a)$ exists, then $\int_a^a f(x)\ dx = 0$.

Example

Evaluate the integral.

1. $\int_0^2 3\ dx$
2. $\int_2^2 (x^2 + 4)\ dx$

Theorem

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Example

Evaluate the integral.

1. $\int_0^2 3\ dx$
2. $\int_2^2 (x^2 + 4)\ dx$

Solution:

1. $\int_0^2 3\ dx = 3(2 - 0) = 6.$
2. $\int_2^2 (x^2 + 4)\ dx = 0.$

Example

If $\int_a^b f(x)\,dx = 4$ *and* $\int_a^b g(x)\,dx = 2$, *then find* $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx$.

Example

If $\int_a^b f(x)\ dx = 4$ *and* $\int_a^b g(x)\ dx = 2$, *then find* $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx$.

Solution:

$$\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx = 3\int_a^b f(x)\ dx - \frac{1}{2}\int_a^b g(x)\ dx = 3(4) - \frac{1}{2}(2) = 11.$$

Example

If $\int_a^b f(x)\,dx = 4$ *and* $\int_a^b g(x)\,dx = 2$, *then find* $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx$.

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Example

Prove that $\int_0^2 (x^3 + x^2 + 2)\,dx \geq \int_0^2 (x^2 + 1)\,dx$ *without evaluating the integrals.*

Example

If $\int_a^b f(x)\,dx = 4$ *and* $\int_a^b g(x)\,dx = 2$*, then find* $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx$*.*

Solution:

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Example

Prove that $\int_0^2 (x^3 + x^2 + 2)\,dx \geq \int_0^2 (x^2 + 1)\,dx$ *without evaluating the integrals.*

Solution:

Remember: Property (4) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 + 1$. We can find that $f(x) - g(x) = x^3 + 1 > 0$ for all $x \in [0, 2]$. This implies that $f(x) - g(x) > 0 \Rightarrow f(x) > g(x)$ and from the theorem, we have

$$\int_0^2 (x^3 + x^2 + 2)\,dx \geq \int_0^2 (x^2 + 1)\,dx.$$

Section 3: The Fundamental Theorem of Calculus

Theorem

Suppose that f is continuous on a closed interval $[a, b]$.

1. *If $F(x) = \int_a^x f(t)\, dt$ for every $x \in [a, b]$, then $F(x)$ is an antiderivative of f on $[a, b]$.*
2. *If $F(x)$ is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.*

Corollary

If F is an antiderivative of f, then

$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Notes:

■ From the previous corollary, the definite integral $\int_a^b f(x)\, dx$ is evaluated by two steps:

Step 1: Find an antiderivative F of the integrand,

Step 2: Evaluate the antiderivative F at upper and lower limits by substituting $x = b$ and $x = a$ into F, then subtracting the latter from the former i.e., calculate $F(b) - F(a)$.

Example

Evaluate the integral.

1. $\int_{-1}^{2} (2x + 1)\ dx$
2. $\int_{0}^{3} (x^2 + 1)\ dx$
3. $\int_{0}^{1} x^2 (x^3 + 1)^4\ dx$
4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x + 1)\ dx$

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4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\ dx$

Solution:

1. $\int_{-1}^{2} (2x+1)\ dx = \left[x^2+x\right]_{-1}^{2} = (4+2) - ((-1)^2+(-1)) = 6 - 0 = 6.$

Example

Evaluate the integral.

1. $\int_{-1}^{2} (2x+1)\, dx$
2. $\int_{0}^{3} (x^2+1)\, dx$
3. $\int_{0}^{1} x^2(x^3+1)^4\, dx$
4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\, dx$

Solution:

1. $\int_{-1}^{2} (2x+1)\, dx = \left[x^2+x\right]_{-1}^{2} = (4+2) - ((-1)^2 + (-1)) = 6 - 0 = 6.$
2. $\int_{0}^{3} (x^2+1)\, dx = \left[\frac{x^3}{3}+x\right]_{0}^{3} = (\frac{27}{3}+3) - 0 = 12.$

Example

Evaluate the integral.

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2. $\int_{0}^{3} (x^2+1)\, dx$
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Solution:

1. $\int_{-1}^{2} (2x+1)\, dx = \left[x^2+x\right]_{-1}^{2} = (4+2) - ((-1)^2+(-1)) = 6-0 = 6.$
2. $\int_{0}^{3} (x^2+1)\, dx = \left[\frac{x^3}{3}+x\right]_{0}^{3} = (\frac{27}{3}+3) - 0 = 12.$
3. $\int_{0}^{1} x^2(x^3+1)^4\, dx = \frac{1}{3}\int_{0}^{1} 3x^2(x^3+1)^4\, dx = \frac{1}{3}\frac{1}{5}\left[(x^3+1)^5\right]_{0}^{1} = \frac{1}{15}\left((1^3+1)^5-(0^3+1)^5\right) = \frac{31}{15}.$

Example

Evaluate the integral.

1. $\int_{-1}^{2} (2x+1)\, dx$
2. $\int_{0}^{3} (x^2+1)\, dx$
3. $\int_{0}^{1} x^2(x^3+1)^4\, dx$
4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\, dx$

Solution:

1. $\int_{-1}^{2} (2x+1)\, dx = \left[x^2+x\right]_{-1}^{2} = (4+2) - ((-1)^2+(-1)) = 6-0 = 6.$
2. $\int_{0}^{3} (x^2+1)\, dx = \left[\frac{x^3}{3}+x\right]_{0}^{3} = (\frac{27}{3}+3) - 0 = 12.$
3. $\int_{0}^{1} x^2(x^3+1)^4\, dx = \frac{1}{3}\int_{0}^{1} 3x^2(x^3+1)^4\, dx = \frac{1}{3}\frac{1}{5}\left[(x^3+1)^5\right]_{0}^{1} = \frac{1}{15}\left((1^3+1)^5-(0^3+1)^5\right) = \frac{31}{15}.$
4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\, dx = \left[-\cos\ x+x\right]_{0}^{\frac{\pi}{2}} = (-\cos\ \frac{\pi}{2}+\frac{\pi}{2}) - (-\cos\ 0+0) = \frac{\pi}{2}+1.$

Example

If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, *find* $\int_{-1}^{2} f(x)\, dx$.

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If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, *find* $\int_{-1}^{2} f(x)\, dx$.

Solution:

Remember: Property (6) If f is integrable on the intervals $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The definition of the function f changes at 0. Since $[-1, 2] = [-1, 0] \cup [0, 2]$, then from the theorem,

$$\begin{aligned} \int_{-1}^{2} f(x)\, dx &= \int_{-1}^{0} f(x)\, dx + \int_{0}^{2} f(x)\, dx \\ &= \int_{-1}^{0} x^2\, dx + \int_{0}^{2} x^3\, dx \\ &= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{2} \\ &= \frac{1}{3} + \frac{16}{4} = \frac{13}{3}. \end{aligned}$$

Mean Value Theorem

Theorem

If f is continuous on a closed interval $[a, b]$*, then there is at least a number* $z \in (a, b)$ *such that*

$$\int_a^b f(x)\,dx = f(z)(b-a).$$

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Find a number z that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = x^2 + 1$ on the interval $[0, 2]$.

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Example

Find a number z that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = x^2 + 1$ on the interval $[0, 2]$.

Solution:
From the theorem,

$$\int_0^2 (1 + x^2)\, dx = (2 - 0)f(z)$$

$$\left[x + \frac{x^3}{3}\right]_0^2 = 2(1 + z^2)$$

$$\frac{14}{3} = 2(1 + z^2)$$

$$\frac{7}{3} = 1 + z^2$$

$$\int_0^2 (1 + x^2)\, dx = \left[x + \frac{x^3}{3}\right]_0^2 = (2 + \frac{8}{3}) - (0) = \frac{14}{3}$$

This implies $z^2 = \frac{4}{3}$, then $z = \pm\frac{2}{\sqrt{3}}$. However, $-\frac{2}{\sqrt{3}} \notin (0, 2)$, so $z = \frac{2}{\sqrt{3}} \in (0, 2)$.

Average Value

Definition

If f is continuous on the interval $[a, b]$, then the average value f_{av} of f on $[a, b]$ is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

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Example

Find the average value of the function $f(x) = x^3 + x - 1$ on the interval $[0, 2]$.

Solution:

$f_{av} = \frac{1}{2-0}\int_0^2 (x^3 + x - 1)\,dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^2 = \frac{1}{2}\Big[(4 + 2 - 2) - (0)\Big] = 2.$

Section 3: The Fundamental Theorem of Calculus

From the Fundamental Theorem, if f is continuous on $[a, b]$ and $F(x) = \int_c^x f(t)\, dt$ where $c \in [a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\, dt = \frac{d}{dx}\Big[F(x) - F(a)\Big] = f(x) \quad \forall x \in [a, b].$$

Theorem

Let f be continuous on an interval I. If h and g are two differentiable functions on an interval J such that $h(J) \subset I$ and $g(J) \subset I$, then the function

$$F(x) = \int_{g(x)}^{h(x)} f(t)\, dt$$

is differentiable on the interval J. Moreover,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))\, h'(x) - f(g(x))\, g'(x) \quad \forall x \in J.$$

Corollary

Let f be continuous on an interval I. If h and g are two differentiable functions on an interval J such that $h(J) \subset I$ and $g(J) \subset I$, then

1. $\frac{d}{dx}\int_a^{h(x)} f(t)\, dt = f(h(x))h'(x) \quad \forall x \in [a, b]$,
2. $\frac{d}{dx}\int_{g(x)}^{a} f(t)\, dt = -f(g(x))g'(x) \quad \forall x \in [a, b]$.

Example

Find the derivative: $\frac{d}{dx}\int_{x}^{\sin x} \frac{1}{1-t^2}\,dt$

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Solution:

$$\begin{aligned}
\frac{d}{dx}\int_{x}^{\sin x} \frac{1}{1-t^2}\, dt &= \frac{1}{1-\sin^2 x}(\cos x) - \frac{1}{1-x^2}(1) \\
&= \frac{\cos x}{\cos^2 x} - \frac{1}{1-x^2} \qquad \text{use the identity } \cos^2 x + \sin^2 x = 1 \\
&= \sec x - \frac{1}{1-x^2}\,.
\end{aligned}$$

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Find the derivative: $\frac{d}{dx}\int_{x}^{\sin x} \frac{1}{1-t^2}\, dt$

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\end{aligned}$$

Example

If $F(x) = (x^2 - 2)\int_{2}^{x} \left(t + 3F'(t)\right) dt$*, find* $F'(2)$.

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If $F(x) = (x^2 - 2)\int_2^x \left(t + 3F'(t)\right) dt$*, find* $F'(2)$.

Solution: Let $J_1(x) = x^2 - 2$ and $J_2(x) = \int_2^x \left(t + 3F'(t)\right) dt$. Then find

$$F'(x) = J_1'(x)J_2(x) + J_1(x)J_2'(x)$$

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Example

If $F(x) = (x^2-2)\int_2^x \left(t + 3F'(t)\right)\ dt$*, find* $F'(2)$*.*

Solution: Let $J_1(x) = x^2 - 2$ and $J_2(x) = \int_2^x \left(t + 3F'(t)\right)\ dt$. Then find

$$F'(x) = J_1'(x)J_2(x) + J_1(x)J_2'(x) \Rightarrow F'(x) = 2x$$

Example

Find the derivative: $\frac{d}{dx}\int_{x}^{\sin x} \frac{1}{1-t^2}\,dt$

Solution:

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$$F'(x) = J_1'(x)J_2(x) + J_1(x)J_2'(x) \Rightarrow F'(x) = 2x\int_2^x \left(t+3F'(t)\right)\,dt$$

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Find the derivative: $\frac{d}{dx}\int_x^{\sin x} \frac{1}{1-t^2}\, dt$

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Letting $x = 2$ gives

$$F'(2) = 4\int_2^2 \left(t + 3F'(t)\right)\, dt + (4-2)\left(2 + 3F'(2)\right)$$

Example

Find the derivative: $\frac{d}{dx}\int_x^{\sin\ x} \frac{1}{1-t^2}\ dt$

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Trapezoidal Rule

Trapezoidal Rule

Let f be continuous on $[a, b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a regular partition of $[a, b]$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\Big] = \frac{b-a}{2n}\sum_{k=0}^{n} m_k\ f(x_k) \qquad (2)$$

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Example

By using the trapezoidal rule with $n = 4$, approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

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By using the trapezoidal rule with $n = 4$, approximate the integral $\int_1^2 \frac{1}{x}\, dx$.

Solution: Find the regular partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.

We divide the interval $[1, 2]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{2-1}{4} = \frac{1}{4}$ as follows:

$x_0 = 1$
$x_1 = 1 + 1(\frac{1}{4}) = 1\frac{1}{4}$
$x_2 = 1 + 2(\frac{1}{4}) = 1\frac{1}{2}$
$x_3 = 1 + 3(\frac{1}{4}) = 1\frac{3}{4}$
$x_4 = 1 + 4(\frac{1}{4}) = 2$

The partition is $P = \{1, 1.25, 1.5, 1.75, 2\}$.

We use the following table to approximate the integral:

x_k	$f(x_k)$	m_k	$m_k f(x_k)$
1	1	1	1
1.25	0.8	2	1.6
1.5	0.6667	2	1.3334
1.75	0.5714	2	1.1428
2	0.5	1	0.5
$\sum_{k=1}^{4} m_k f(x_k)$			5.5762

Hence,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\Big] = \frac{b-a}{2n}\sum_{k=0}^{n} m_k\ f(x_k)$$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{8}\left[5.5762\right] = 0.697.$$

Simpson's Rule

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Let f be continuous on $[a, b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a regular partition of $[a, b]$ where n is even, then

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{3n}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big] = \frac{(b-a)}{3n}\sum_{k=0}^{n} m_k\, f(x_k) \tag{3}$$

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Example

By using Simpson's rule with $n = 4$, approximate the integral $\int_1^3 \sqrt{x^2+1}\,dx$.

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Solution: Find the regular partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.
We divide the interval $[1, 3]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$ as follows:

$x_0 = 1$
$x_1 = 1 + 1(\frac{1}{2}) = 1\frac{1}{2}$
$x_2 = 1 + 2(\frac{1}{2}) = 2$
$x_3 = 1 + 3(\frac{1}{2}) = 2\frac{1}{2}$
$x_4 = 1 + 4(\frac{1}{2}) = 3$

The partition is $P = \{1, 1.5, 2, 2.5, 3\}$.

We use the following table to approximate the integral:

x_k	$f(x_k)$	m_k	$m_k f(x_k)$
1	1.4142	1	1.4142
1.5	1.8028	4	7.2112
2	2.2361	2	4.4722
2.5	2.6926	4	10.7704
3	3.1623	1	3.1623
$\sum_{k=1}^{4} m_k f(x_k)$			27.0302

Hence,

$$\int_a^b f(x)\, dx \approx \frac{(b-a)}{3n}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big] = \frac{(b-a)}{3n}\sum_{k=0}^{n} m_k\ f(x_k)$$

$$\int_1^3 \sqrt{x^2+1}\, dx \approx \tfrac{2}{12}\,[27.0302] = 4.5050.$$