Ch.23: Continuous Charge Distributions and Gauss's Law

Physics 104: Electricity and Magnetism

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Outline

Remember From Previous Chapters

Classical Mechanics

- Equations of motion:

$$\vec{\boldsymbol{v}}_f = \vec{\boldsymbol{v}}_i + \vec{\boldsymbol{a}}t$$

$$\vec{\boldsymbol{r}}_f = \vec{\boldsymbol{r}}_i + \vec{\boldsymbol{v}}_i t + \frac{1}{2}\vec{\boldsymbol{a}}t^2$$

$$\vec{\boldsymbol{v}}_f^2 = \vec{\boldsymbol{v}}_i^2 + 2\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{r}}_f - \vec{\boldsymbol{r}}_i\right)$$

- Newton's Second Law:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

- Work-Energy Theorem:

$$W = \Delta K$$

$$W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{d}}$$

$$K = \frac{1}{2}m\vec{\boldsymbol{v}}^2$$

Electric Field

- Coulomb's Law:

$$\vec{\boldsymbol{F}}_e = k_e \frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$$

- Electric Field:

$$\vec{\boldsymbol{E}} = \frac{\vec{\boldsymbol{F}}_e}{q_0} = k_e \frac{q}{r^2}\hat{\boldsymbol{r}}$$

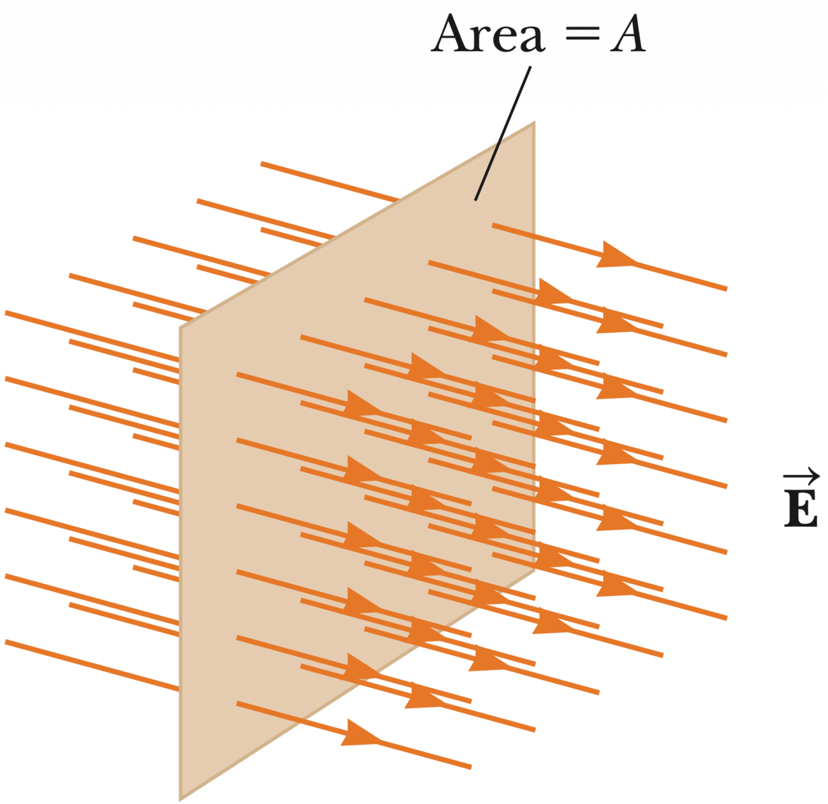
$$\vec{\boldsymbol{a}} = \left(\frac{q}{m}\right)\vec{\boldsymbol{E}}$$

1. Electric Flux

2. Gauss's Law

3. Application of Gauss's Law to Various Charge Distributions

1.1 Definition of Electric Flux

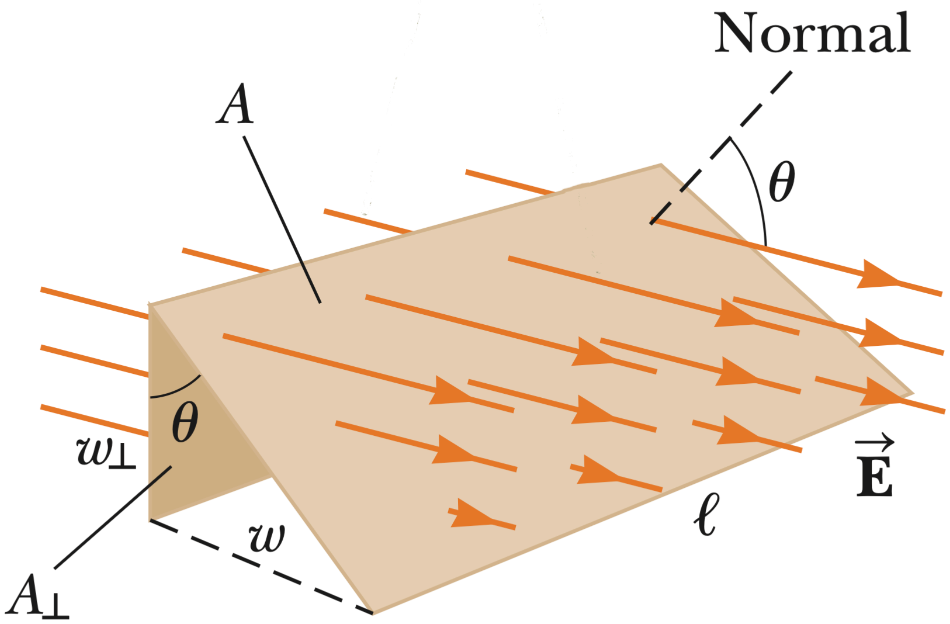


Electric Flux Φ_E measures the amount (or "flow") of electric field E passing through a given surface A.

$$\Phi_E = EA$$

- E is perpendicular to the surface.
- Φ_E has a unit of **N·m²/C**.

1.1 Definition of Electric Flux

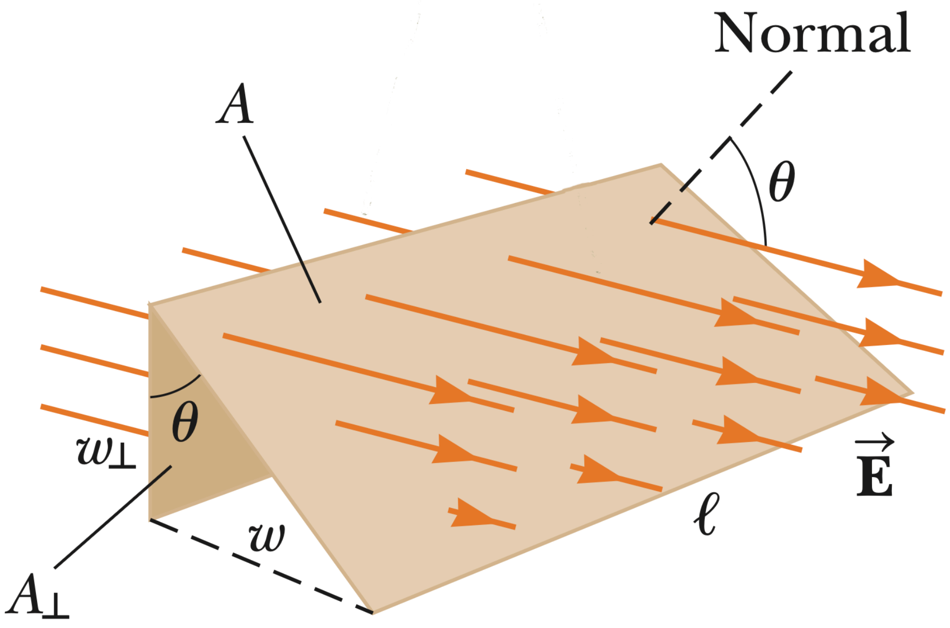


- The number of field lines passing through the area $A_\perp$ *equals* the number of field lines passing through area A at an angle θ to the normal.#pause
- Therefore,

$$\Phi_E = EA_\perp = EA\cos\theta = \vec{E}\cdot\vec{A}$$

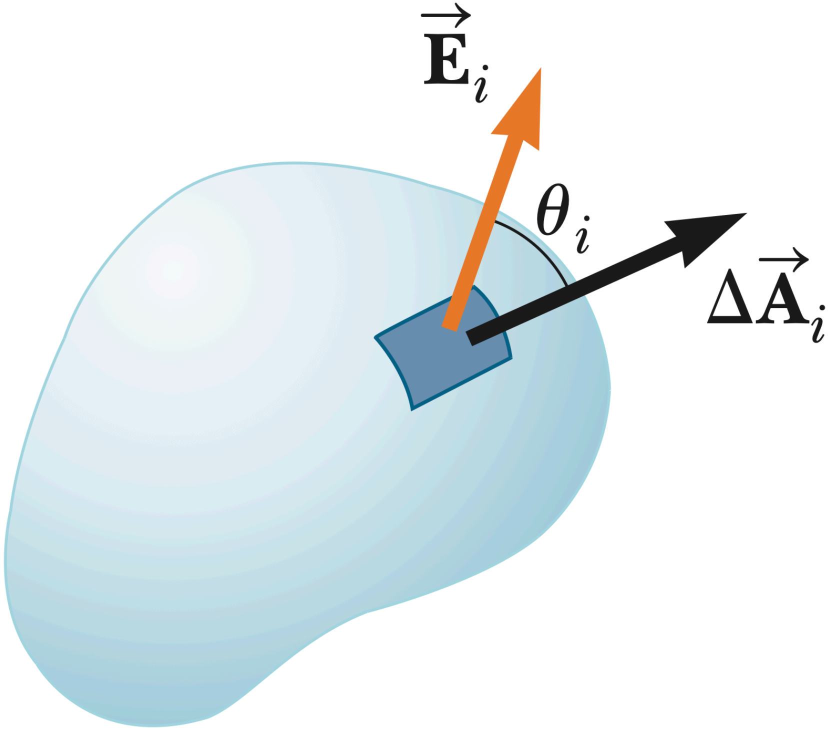
- $\vec{A}$ is a vector whose magnitude is the area A and direction is normal (perpendicular) to the surface.
- The flux Φ_E is a *scalar* quantity.

1.1 Definition of Electric Flux



- Therefore, we conclude the following:
- The flux through a surface of fixed area (A) has a maximum value EA when the surface is perpendicular to the field, and
- a minimum value of zero when the surface is parallel to the field.
- The flux vector $\vec{\boldsymbol{\Phi}}_E$ can be positive or negative, depending on the angle θ between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{A}}$.

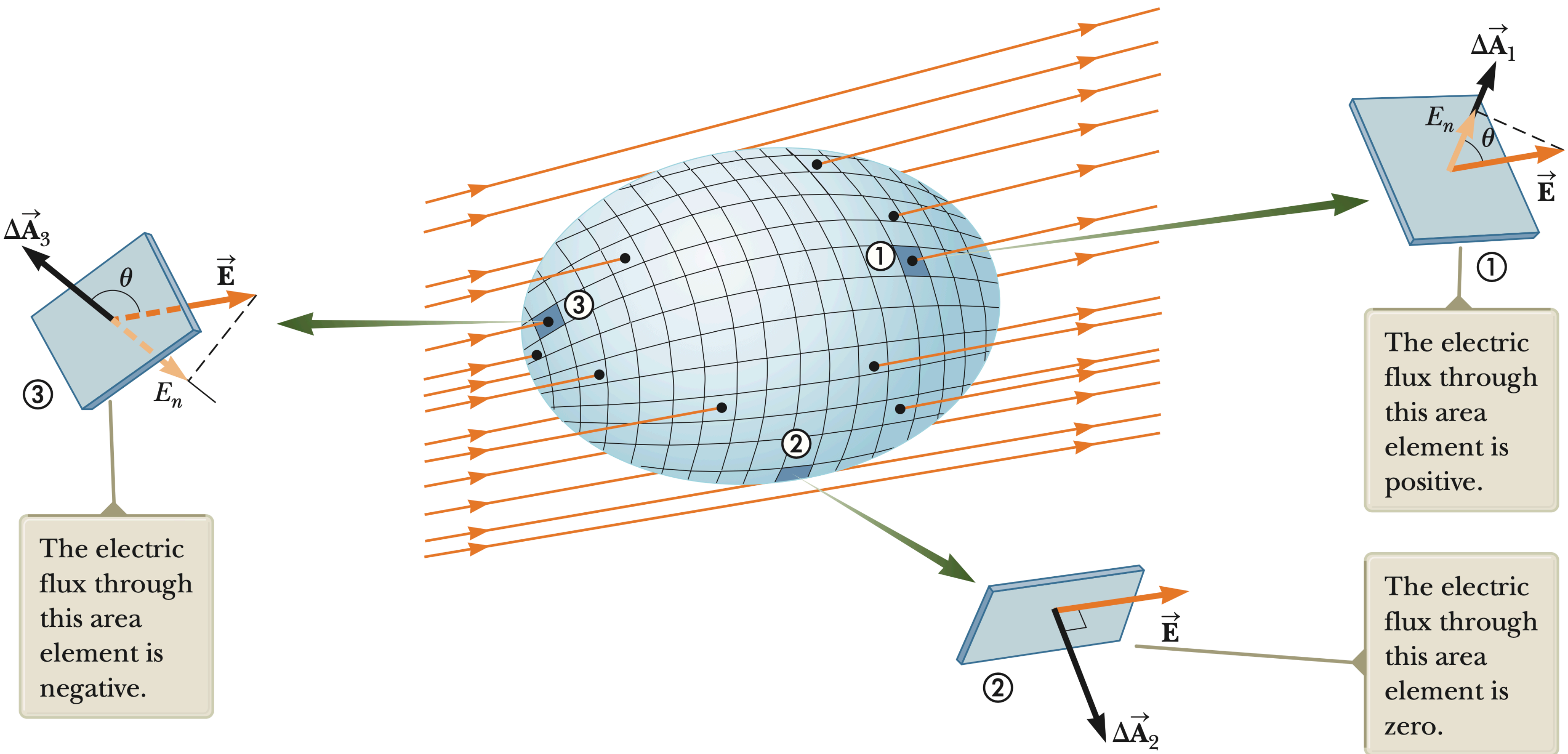
1.2 Electric Flux through a Curved and large Surface



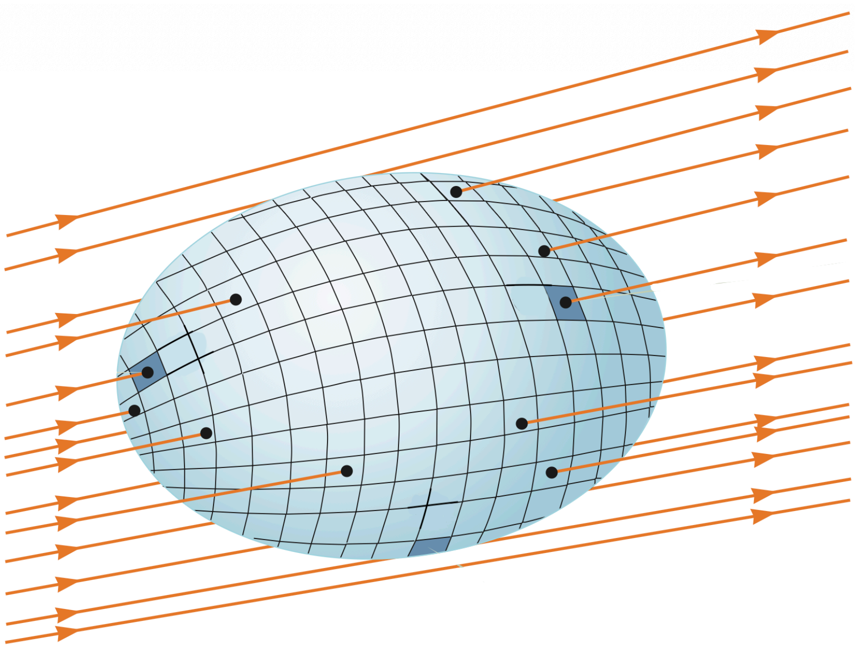
- The electric field $\vec{\boldsymbol{E}}$ may vary in magnitude and direction over a curved surface.
- The area vector $\vec{\boldsymbol{A}}$ may also vary in direction over the surface.
- Therefore, we divide the surface into small elements of area $\mathrm{d}\vec{\boldsymbol{A}}$ and calculate the flux through each element.

$$\Phi_{E,i} = \vec{\boldsymbol{E}}_i \cdot \Delta\vec{\boldsymbol{A}}_i = E_i \Delta A_i \cos\theta_i$$

1.2 Electric Flux through a Curved and large Surface



1.2 Electric Flux through a Curved and large Surface



- The total electric flux Φ_E through the entire surface is the sum of all small contributions,

$$\Phi_E \approx \sum_i \vec{E}_i \cdot \Delta\vec{A}_i$$

- In the limit as the area elements become infinitesimally small, the sum becomes a surface integral:

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A}$$

1.3 Example

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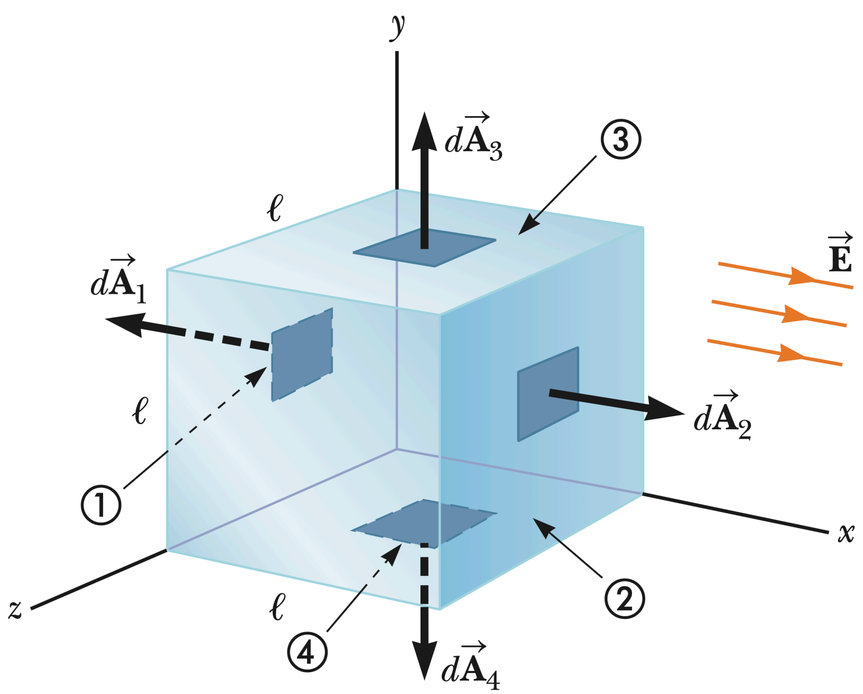
Example 1.1

Consider a uniform electric field $\vec{E}$ oriented in the x direction in empty space. A cube of edge length l is placed in the field, oriented as shown in the Figure. Find the net electric flux through the surface of the cube.



1.3 Example

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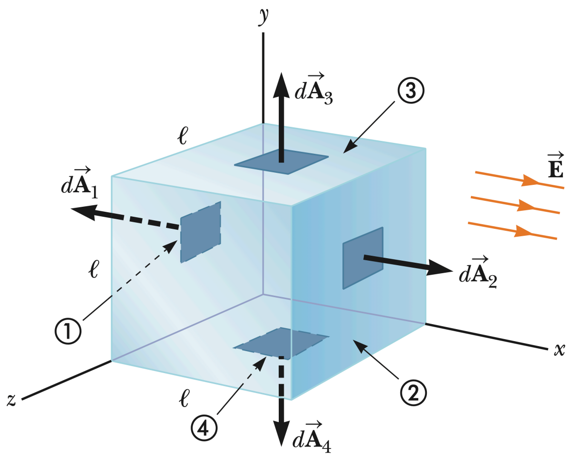


Solution 1.1

$$\Phi_E = \Phi_{E,1} + \Phi_{E,2} + \Phi_{E,3} + \Phi_{E,4}$$

$$\Phi_{E,1} = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E \cos 180° \oint \mathrm{d}A = -EA = -El^2$$

$$\Phi_{E,2} = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E \cos 0° \oint \mathrm{d}A = EA = El^2$$

$$\Phi_{E,3} = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E \cos 90° \oint \mathrm{d}A = 0$$

$$\Phi_{E,4} = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E \cos 270° \oint \mathrm{d}A = 0$$

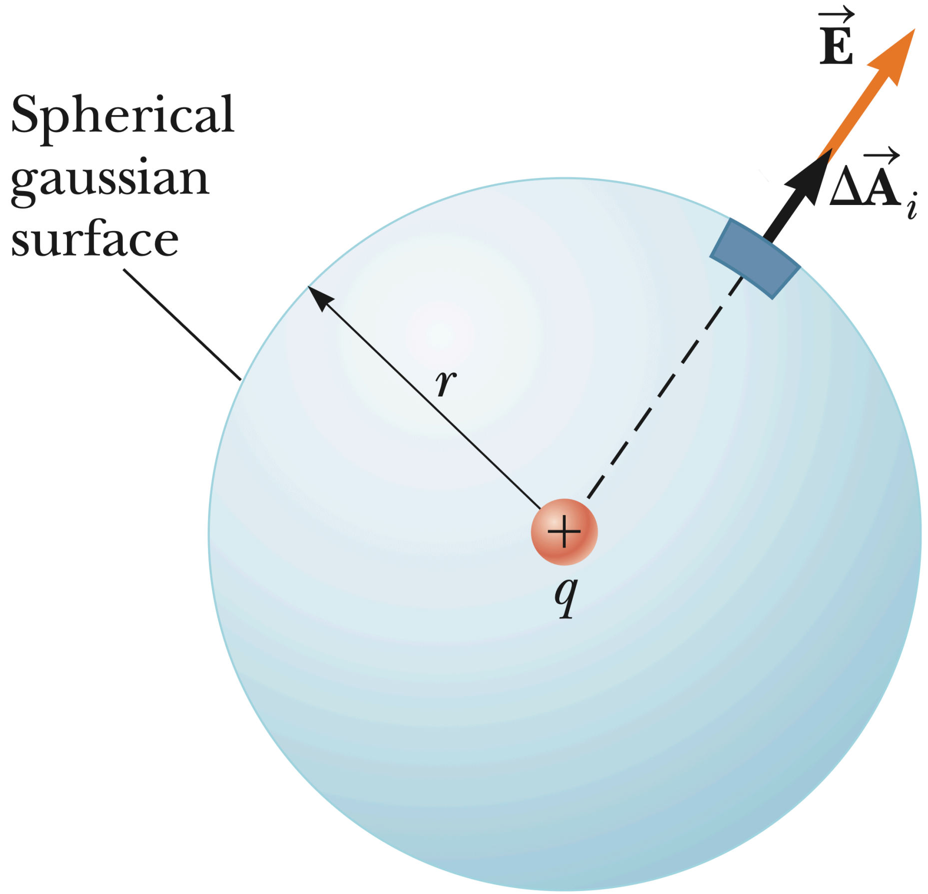
$$\implies \quad \Phi_E = -El^2 + El^2 + 0 + 0 = 0$$

1. Electric Flux

2. Gauss's Law

3. Application of Gauss's Law to Various Charge Distributions

2.1 Derivation of Gauss's Law



Consider a point charge q located at the center of a spherical surface of radius r, known as a **gaussian surface**.

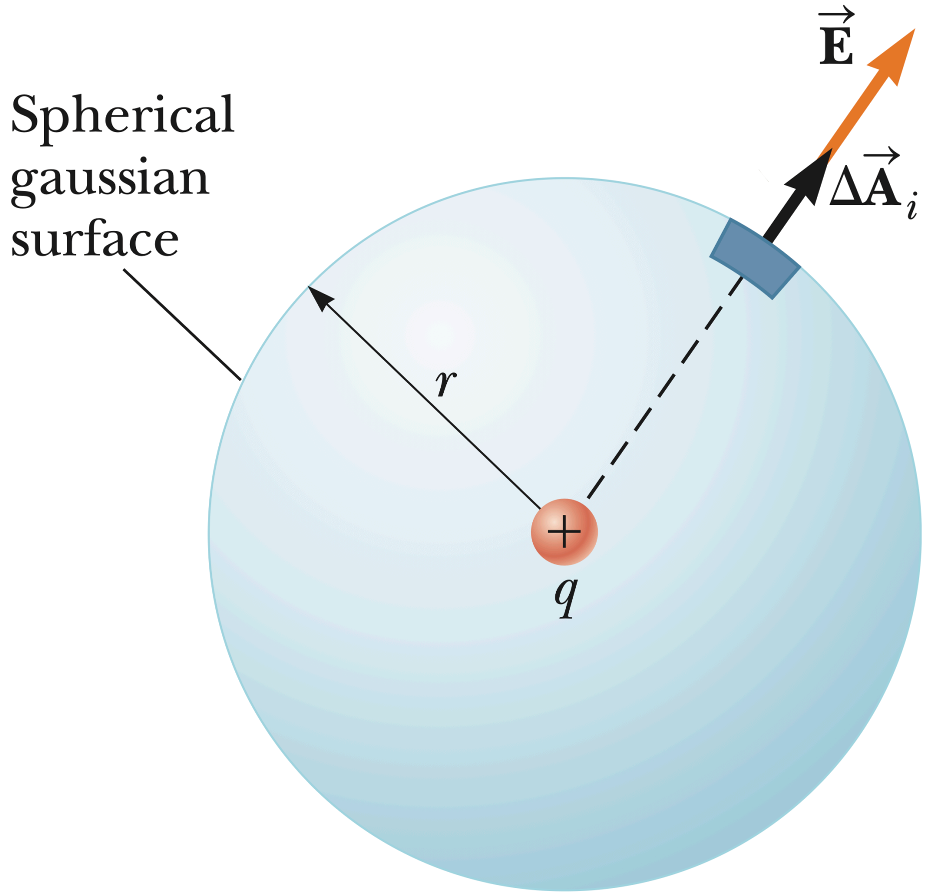
The electric field at every point on the surface has a magnitude

$$E = k_e \frac{q}{r^2}$$

The area of the spherical surface is

$$A = 4\pi r^2$$

2.1 Derivation of Gauss's Law



The electric flux through the spherical surface is

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = EA$$

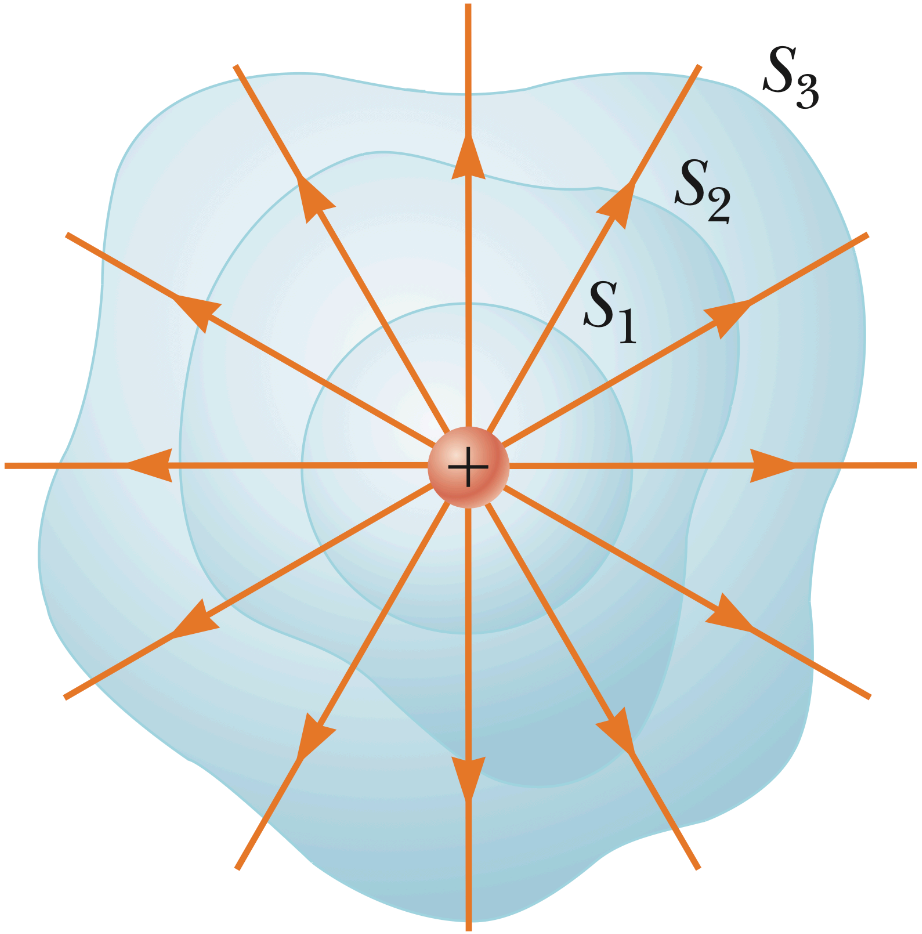
$$= \left(k_e \frac{q}{r^2}\right)(4\pi r^2) = 4\pi k_e q$$

Since $k_e = \frac{1}{4\pi\varepsilon_0}$, we have

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q}{\varepsilon_0}$$

Notice that Φ_E does not depend on r

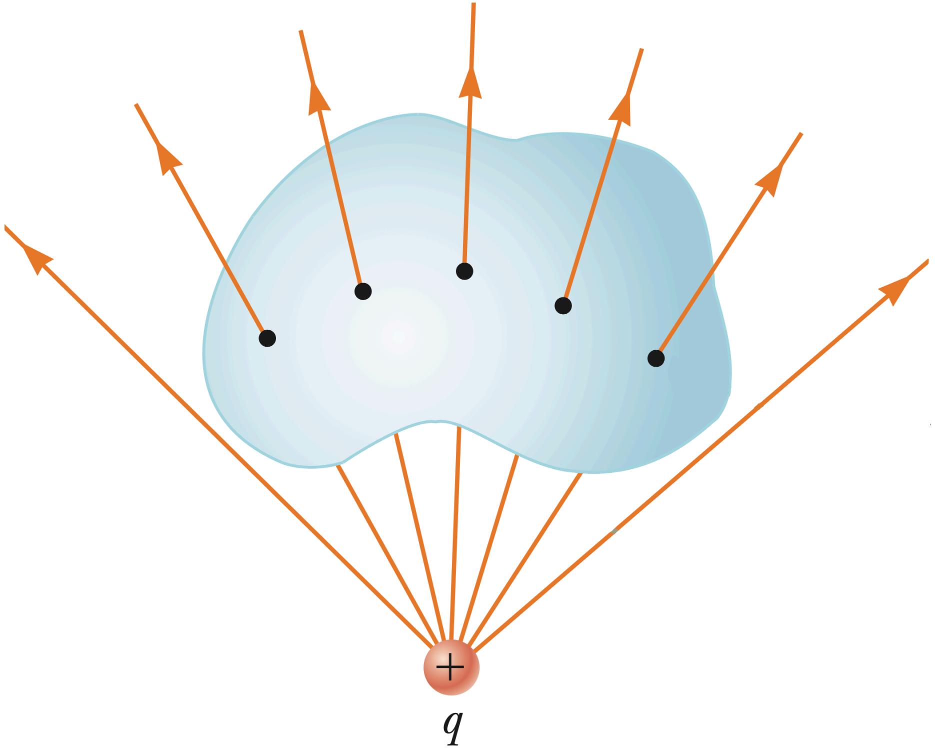
2.1 Derivation of Gauss's Law



- Every field line that passes through S_1 also passes through the nonspherical surfaces S_2 and S_3. Therefore,
- The net flux through any closed surface surrounding a point charge q is independent of the shape of that surface.

$$\Phi_{ES_1} = \Phi_{ES_2} = \Phi_{ES_3} = \frac{q}{\varepsilon_0}$$

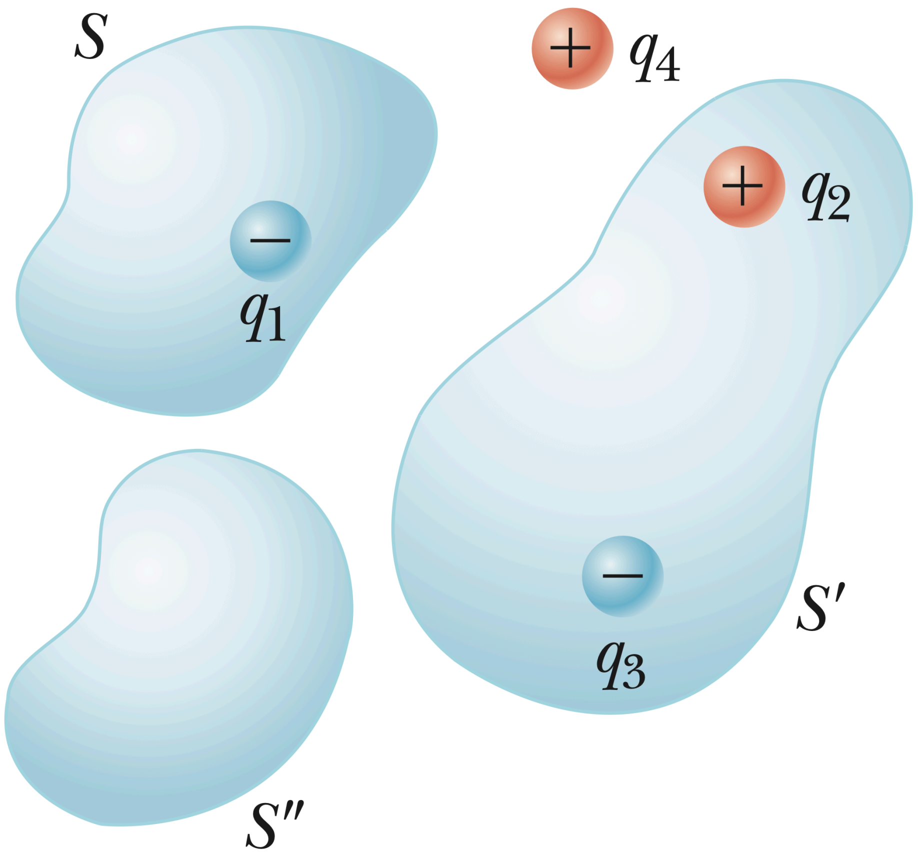
2.1 Derivation of Gauss's Law



- Any electric field line entering the surface leaves the surface at another point.
- The number of electric field lines entering the surface equals the number leaving the surface.
- Therefore, the net electric flux through a closed surface that surrounds no charge is zero.

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = 0$$

2.1 Derivation of Gauss's Law



- The net electric flux through any closed surface depends only on the charge inside that surface.

$$\Phi_{\text{ES}} = \frac{q_1}{\varepsilon_0}$$

$$\Phi_{\text{ES'}} = \frac{q_2 + q_3}{\varepsilon_0}$$

$$\Phi_{\text{ES''}} = 0$$

2.1 Derivation of Gauss's Law

Gauss's Law states that the *net* electric flux Φ_E through any closed surface is equal to the *net* charge q_{in} *inside* that surface divided by the permittivity of free space ε_0.

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = \frac{q_{\text{in}}}{\varepsilon_0}$$

where $\vec{\boldsymbol{E}}$ is the electric field at any point on the closed surface.

2.1 Derivation of Gauss's Law

Example 2.2

A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if:
(A) the charge is tripled,
(B) the radius of the sphere is doubled,
(C) the surface is changed to a cube, and
(D) the charge is moved to another location inside the surface.

2.1 Derivation of Gauss's Law

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Solution 2.2

(A) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(B) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(C) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(D) The flux does not change when the charge is moved to another location inside the surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

1. Electric Flux

2. Gauss's Law

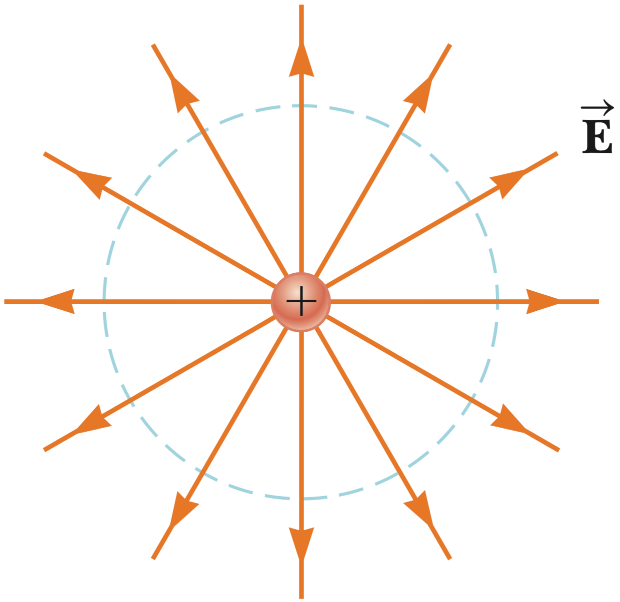
3. Application of Gauss's Law to Various Charge Distributions

3.1 Why to use Gauss's Law?

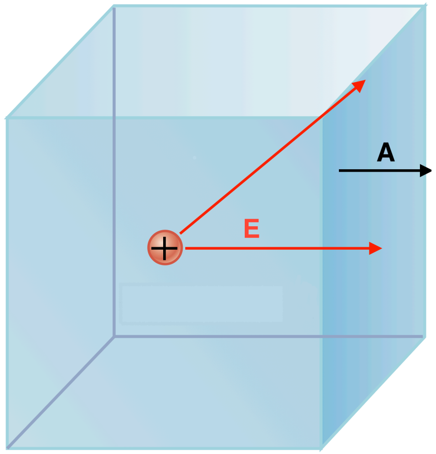
- Gauss's law is a powerful tool for calculating electric fields when the charge distribution has enough degree of symmetry.
- It is often much easier to use Gauss's law than to apply Coulomb's law and perform complex integrations.
- Gauss's law is useful when at least one of four conditions is satisfied for the chosen gaussian surface.

3.2 Four Conditions to Use Gauss's Law

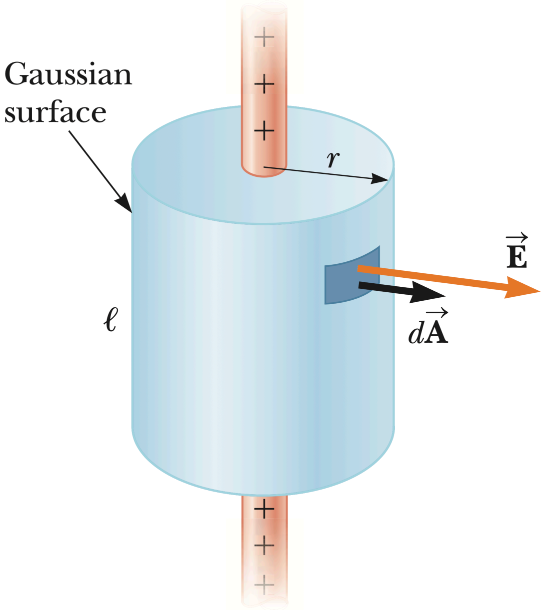
1. The value of the electric field can be argued by symmetry to be **constant** over the portion of the surface (S).



E is constant at S ✓



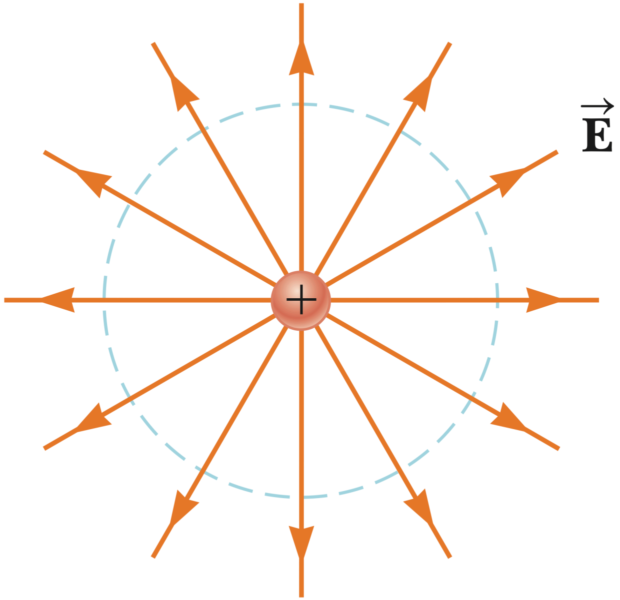
✗ Bad choice, since E is *not* constant over the surface ✗



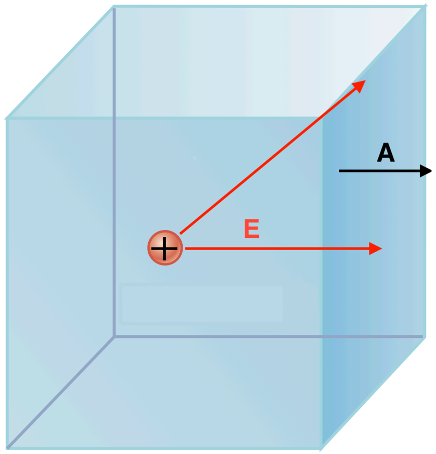
E is constant at S ✓

3.2 Four Conditions to Use Gauss's Law

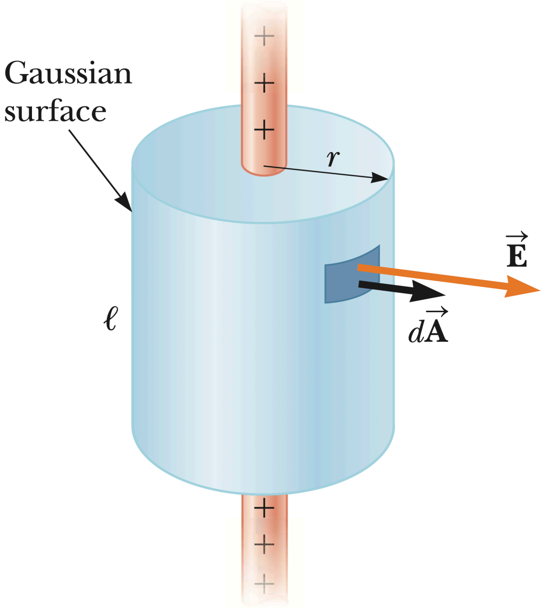
2. The dot product between $\vec{E}$ and $\mathrm{d}\vec{A}$ can be expressed as a simple algebraic product $\vec{E} \cdot \mathrm{d}\vec{A} = E\,\mathrm{d}A$, where $\vec{E}$ and $\mathrm{d}\vec{A}$ are parallel.



$\vec{E} \cdot \mathrm{d}\vec{A} = E\,\mathrm{d}A$ ✓



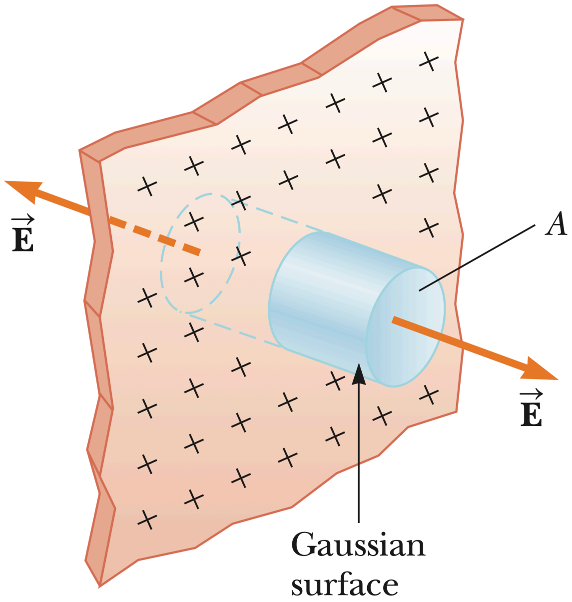
✗ Bad choice of Gaussian surface since E is not generally parallel to $\mathrm{d}\vec{A}$ ✗



$\vec{E} \cdot \mathrm{d}\vec{A} = E\,\mathrm{d}A$ or 0 ✓

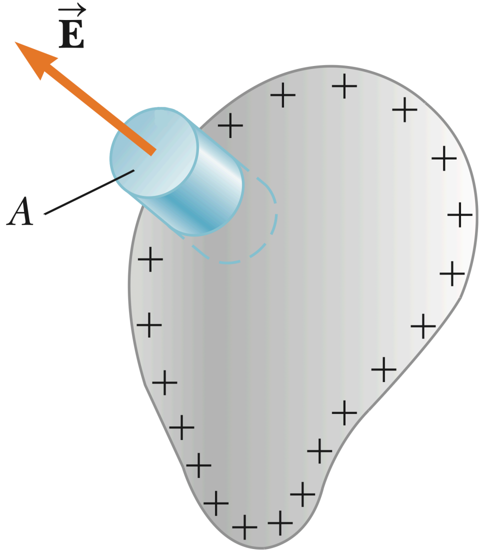
3.2 Four Conditions to Use Gauss's Law

3. The dot product is zero because $\vec{E}$ and $d\vec{A}$ are perpendicular.



$\vec{E} \perp d\vec{A}$ at the curved surface.

4. The electric field is zero over the portion of the surface.



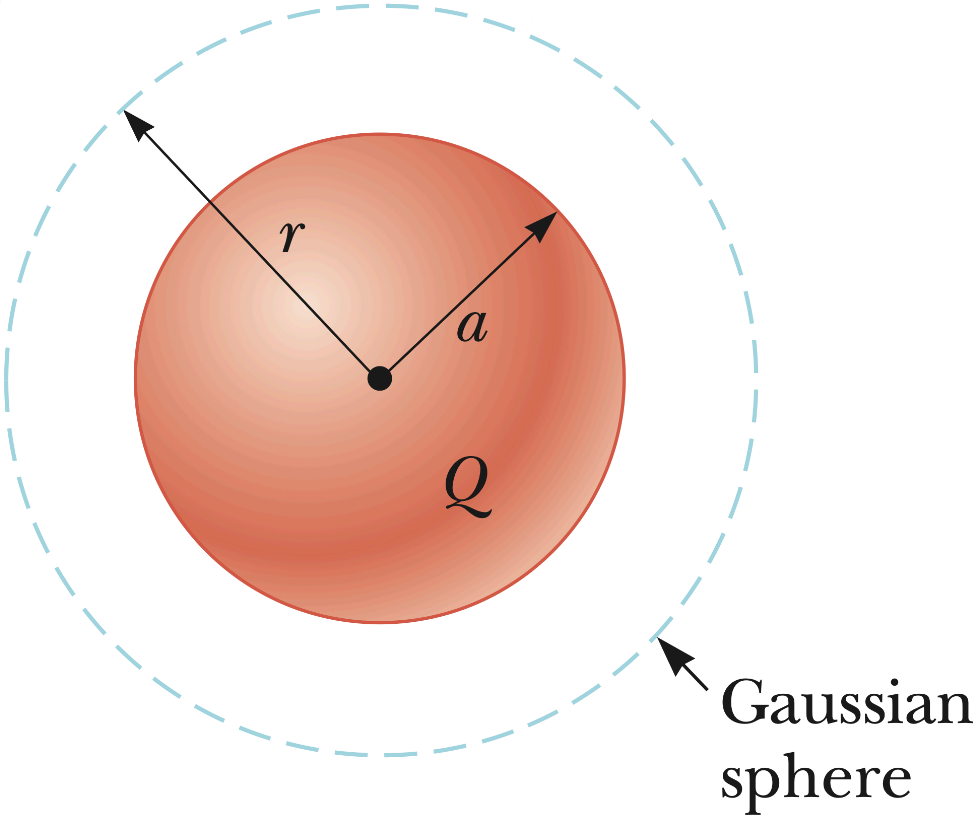
Arbitrary shaped conductor with zero electric field at the curved surface.

3.3 Examples

Example 3.3

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q.

(A) Calculate the magnitude of the electric field at a point outside the sphere.



3.3 Examples

Solution 3.3

Choosing spherical gaussian surface and using Gauss's law, we have:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = \frac{q_{\text{in}}}{\varepsilon_0}$$

Since $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{A}}$ are parallel vectors and $q_{\text{in}} = Q$, we get:

$$\oint E \cos 0^\circ \, \mathrm{d}A = E \oint \mathrm{d}A = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

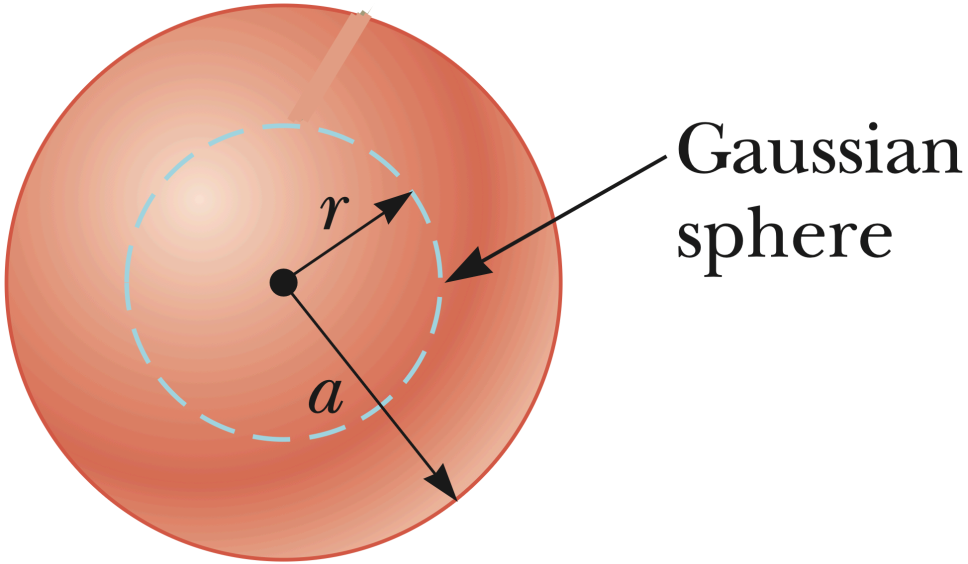
Solving for E, we obtain:

$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$$

$$\Longrightarrow \quad E = k_e \frac{Q}{r^2} \qquad (\text{for } r > a)$$

3.3 Examples

(B) Find the magnitude of the electric field at a point inside the sphere.



3.3 Examples

Solution 3.4

Similar to part (A), we apply Gauss's law:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = E(4\pi r^2) = \frac{q_{\text{in}}}{\varepsilon_0}$$

Solving for E,

$$E = \frac{q_{\text{in}}}{4\pi\varepsilon_0 r^2}$$

Since $r < a$, the internal charge $q_{\text{in}} < Q$. Therefore, to find q_{in}, we multiply the charge density ρ by the volume of a sphere of radius r:

$$q_{\text{in}} = \rho\left(\frac{4}{3}\pi r^3\right) = \left(\frac{Q}{\frac{4}{3}\pi a^3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{a^3}$$

Notice that at $r = a$, we have $q_{\text{in}} = Q$ as expected.

3.3 Examples

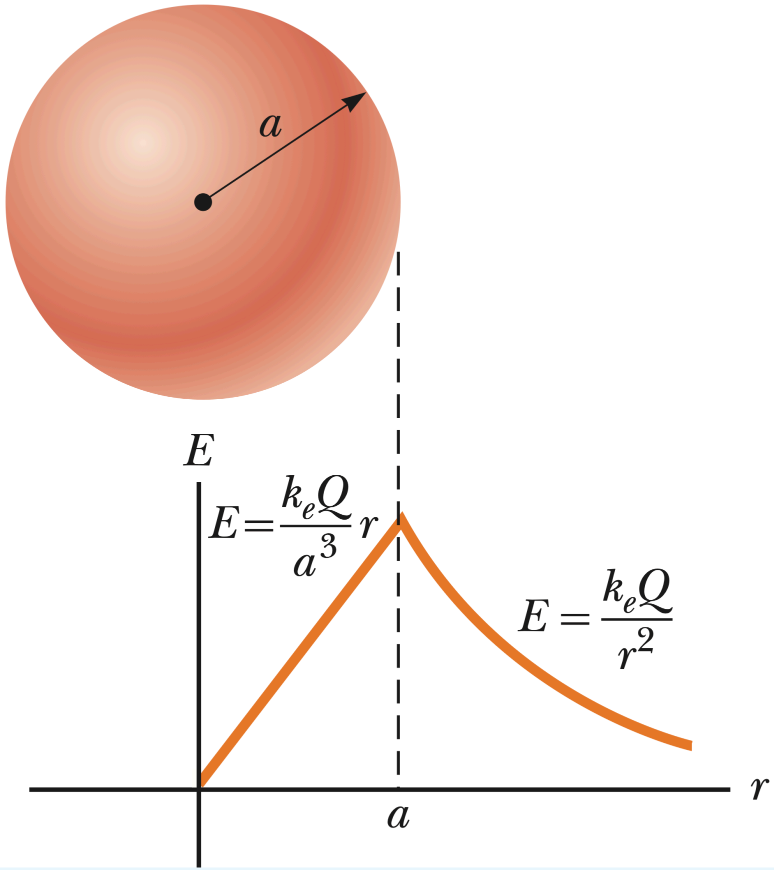
Substituting q_{in} into the expression for E, we get:

$$E = \frac{Q\ r^3/a^3}{4\pi\varepsilon_0 r^2} = \left(\frac{Q}{4\pi\varepsilon_0 a^3}\right) r$$

Therefore,

$$\implies \quad E = k_e \frac{Q}{a^3} r \qquad (\text{for } r < a)$$

3.3 Examples



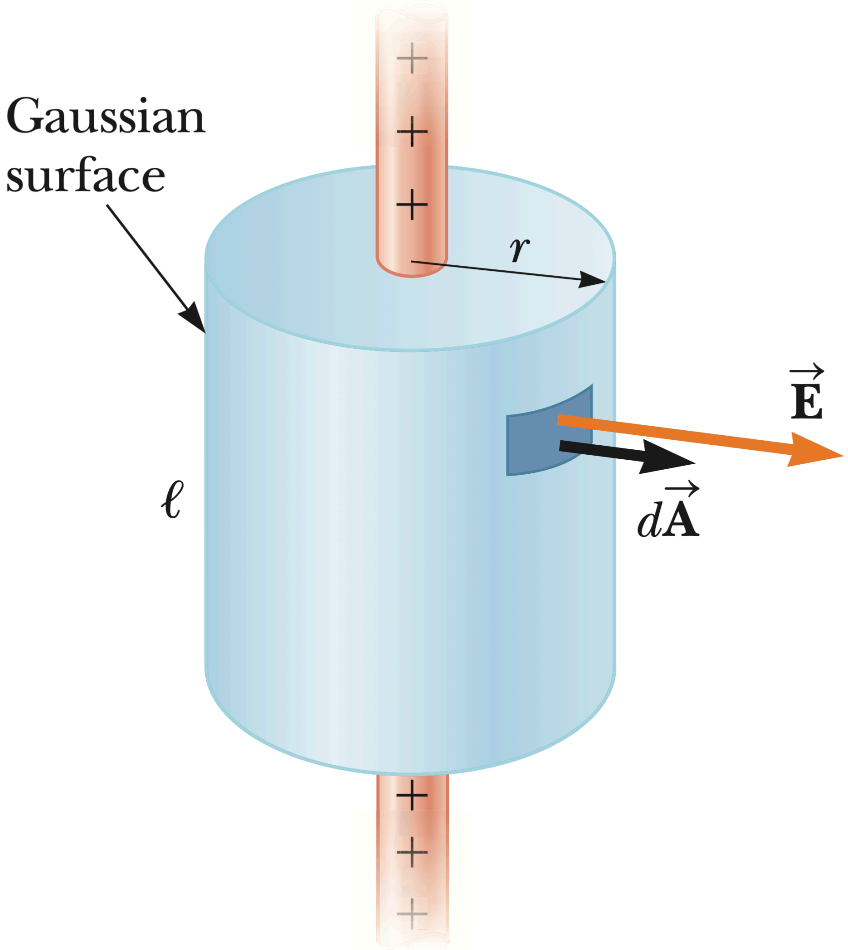
Therefore, the electric field inside the sphere increases linearly with distance r until it reaches its maximum value at the surface of the sphere ($r = a$), then decreases as $1/r^2$ for points outside the sphere.

Additionally, the two expressions for E at $r = a$ are *equal*, confirming the continuity of the electric field at the surface of the sphere.

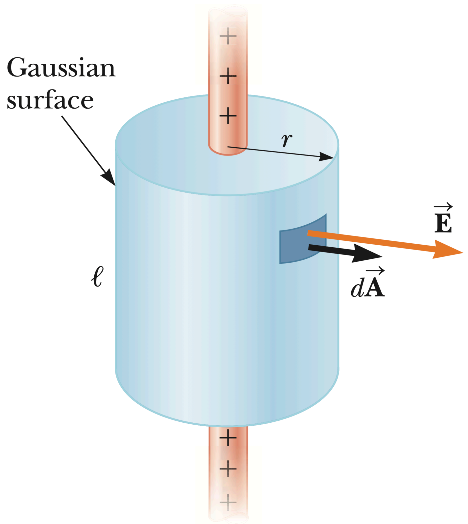
3.3 Examples

Example 3.5

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ



3.3 Examples



Solution 3.5

- Using Gauss's law, we choose a cylindrical gaussian surface of radius r and length l coaxial with the line of charge.
- The electric field $\vec{\boldsymbol{E}}$ is radial and has the same magnitude at every point on the curved surface of the cylinder.
- The area vector $\mathrm{d}\vec{\boldsymbol{A}}$ is also radial on the curved surface, so $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{A}}$ are parallel vectors.
- On the two flat end caps of the cylinder, $\vec{\boldsymbol{E}}$ is perpendicular to $\mathrm{d}\vec{\boldsymbol{A}}$, so there is no flux through these surfaces.

3.3 Examples

Therefore, the total flux through the cylindrical surface is given by:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = E \oint \mathrm{d}A = EA = \frac{q_{\text{in}}}{\varepsilon_0}$$

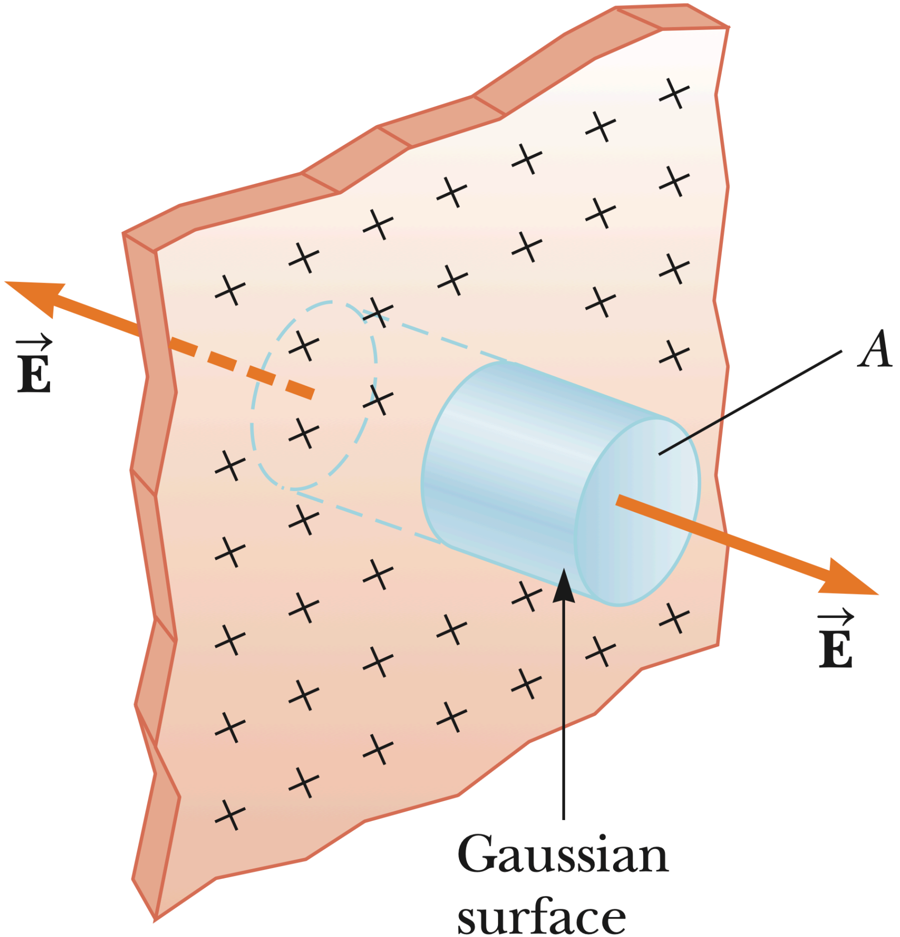
The area of the curved surface is $A = 2\pi r L$, and the charge enclosed by the gaussian surface is the charge density times length ($q_{\text{in}} = \lambda L$). Substituting these expressions into Gauss's law, we have:

$$E(2\pi r L) = \frac{\lambda L}{\varepsilon_0} \implies \quad E = \frac{\lambda}{2\pi\varepsilon_0 r} = 2k_e \frac{\lambda}{r}$$

3.3 Examples

Example 3.6

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.



3.3 Examples

Solution 3.6

- We use a cylindrical gaussian surface (or a cubic surface).
- The electric field $\vec{\boldsymbol{E}}$ is perpendicular to the plane and has the same magnitude at every point on the two flat surfaces.
- $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{A}}$ are parallel vectors at the two flat surfaces.
- On the curved surface, $\vec{\boldsymbol{E}}$ is perpendicular to $\mathrm{d}\vec{\boldsymbol{A}}$, so there is no flux through this surface.
- Therefore, the total flux through the two sides of the cylinder is given by:

$$2\oint \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{A}} = 2E\oint \mathrm{d}A = 2EA = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

3.3 Examples

Solving for E, we get:

$$\Longrightarrow \quad E = \frac{\sigma}{2\varepsilon_0}$$

Notice that the electric field due to an infinite plane of charge is constant and does *not* depend on the distance from the plane.

3.3 Examples

Example 3.7

Explain why Gauss's law cannot be used to calculate the electric field near an electric dipole, a charged disk, or a triangle with a point charge at each corner.

Solution 3.7

The charge distributions of all these configurations do not have sufficient symmetry to make a practical use of Gauss's law.

Suggested Problems

10, 11, 13, 14, 15, 16, 18, 19, 24, 27, 29, 33, 34, 37, 38

Book: Serway, R. A., & Jewett, J. W. (2018). Physics for Scientists and Engineers (10th ed.)

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