

Integral Calculus

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Chapter 1: The Indefinite Integrals

Main Contents.

- Antiderivatives.
- Indefinite Integrals.
- Basic Integration Rules.
- Properties of Indefinite Integrals.
- Integration By Substitution.

Section 1: Antiderivatives

■ **Exercise.** Find the derivative of the given function.

(1) $F(x) = x^3 + x^2 + 1$

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Is the function $H(x) = x^2 - \sqrt[3]{2}$ an antiderivative of the function f ?

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Yes, $H'(x) = 2x = f(x)$.

Theorem

If functions F and G are antiderivatives of f on an interval I , there exists a constant c such that $G(x) = F(x) + c$ OR $G(x) - F(x) = c$ for every $x \in I$.

In the example above, we have for example

$$G(x) - F(x) = x^2 + 2 - x^2 = 2$$

Note. For the example above, the function $F(x) = x^2 + c$ is the general form of the antiderivatives (the family) of the function $f(x) = 2x$.

Section 2: Indefinite Integrals

Definition

Let f be a continuous function on an interval I . The indefinite integral of f is the general antiderivative of f on I :

$$\int f(x) dx = F(x) + c.$$

The function f is called the integrand, the symbol \int is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Note. From the example $F'(x) = \frac{d}{dx}(x^2 + 3x) = 2x + 3 = f(x)$, we say that the function $F(x)$ is an antiderivative of $f(x)$. So,

$$\int (2x + 3) dx = \underbrace{x^2 + 3x}_{F(x)} + c.$$

Properties of Indefinite Integrals

Theorem

Assume f and g have antiderivatives on an interval I , then

① $\frac{d}{dx} \int f(x) dx = f(x).$

③ $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$

② $\int \frac{d}{dx}(F(x)) dx = F(x) + c.$

④ $\int kf(x) dx = k \int f(x) dx$, where k is a constant.

Section 2: Indefinite Integrals

Integration as an Inverse Process of Differentiation

Rule 1. Power of x .

$$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1 .$$

Special case. For $n = 0$, we have $\int 1 dx = x + c$.

From this, $\int 2 dx = 2x + c$ and $\int 3 dx = 3x + c$ etc.

Example

Evaluate the integral.

① $\int x dx$

② $\int x^3 dx$

Note

Note that **Rule 1** cannot be applied for $n = -1$.

For this value, the formula gives

$$\int x^{-1} dx = \frac{x^0}{0} = \infty .$$

Section 2: Indefinite Integrals

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Example

Evaluate the integral.

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Solution:

① $\int x dx = \frac{x^2}{2} + c$

Section 2: Indefinite Integrals

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Evaluate the integral.

① $\int x dx$

② $\int x^3 dx$

Solution:

① $\int x dx = \frac{x^2}{2} + c$

② $\int x^3 dx = \frac{x^4}{4} + c$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

1 $\int 4x \, dx$

2 $\int 7x^3 \, dx$

3 $\int \frac{1}{x^2} \, dx$

4 $\int \frac{1}{\sqrt{x}} \, dx$

Section 2: Indefinite Integrals

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Solution:

$$① \int 4x \, dx = 4 \int x \, dx = 4 \frac{x^2}{2} + c = 2x^2 + c$$

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$$③ \int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

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Remember. $x^{-n} = \frac{1}{x^n}$

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$$④ \int \frac{1}{\sqrt{x}} \, dx = \int \frac{1}{x^{\frac{1}{2}}} \, dx = \int x^{-\frac{1}{2}} \, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$-\frac{1}{2} + \frac{1}{1} = \frac{-1+2}{2} = \frac{1}{2}$$

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Remember.

$$(1) \sqrt{x} = x^{\frac{1}{2}} \quad \text{and} \quad \sqrt[m]{x^n} = x^{\frac{n}{m}}$$

$$(2) \frac{a}{b} \pm \frac{c}{d} = \frac{a \times d \pm c \times b}{b \times d}$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

1 $\int (x + 1) dx$

2 $\int (4x^3 + 2x^2 + 1) dx$

3 $\int (x^2 - \frac{1}{x^3}) dx$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$① \int (x + 1) dx$$

$$② \int (4x^3 + 2x^2 + 1) dx$$

$$③ \int \left(x^2 - \frac{1}{x^3}\right) dx$$

Solution:

$$① \int (x + 1) dx = \int x dx + \int 1 dx = \frac{x^2}{2} + x + c$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$\textcircled{1} \int (x + 1) dx$$

$$\textcircled{2} \int (4x^3 + 2x^2 + 1) dx$$

$$\textcircled{3} \int \left(x^2 - \frac{1}{x^3}\right) dx$$

Solution:

$$\textcircled{1} \int (x + 1) dx = \int x dx + \int 1 dx = \frac{x^2}{2} + x + c$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

$$\textcircled{2} \int (4x^3 + 2x^2 + 1) dx = \int 4x^3 dx + \int 2x^2 dx + \int 1 dx = \frac{4x^4}{4} + \frac{2}{3}x^3 + x + c = x^4 + \frac{2}{3}x^3 + x + c.$$

Section 2: Indefinite Integrals

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$$\textcircled{2} \int (4x^3 + 2x^2 + 1) dx = \int 4x^3 dx + \int 2x^2 dx + \int 1 dx = \frac{4x^4}{4} + \frac{2}{3}x^3 + x + c = x^4 + \frac{2}{3}x^3 + x + c.$$

$$\textcircled{3} \int \left(x^2 - \frac{1}{x^3}\right) dx = \int x^2 dx - \int x^{-3} dx = \frac{x^3}{3} - \frac{x^{-2}}{-2} + c = \frac{x^3}{3} + \frac{1}{2x^2} + c.$$

Section 2: Indefinite Integrals

Rule 2. Trigonometric Functions.

- $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + c$
- $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int -\sin x \, dx = \cos x + c$ OR $\int \sin x \, dx = -\cos x + c$
- $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + c$
- $\frac{d}{dx}(\cot x) = -\csc^2 x \Rightarrow \int -\csc^2 x \, dx = \cot x + c$ OR $\int \csc^2 x \, dx = -\cot x + c$
- $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + c$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x \Rightarrow \int -\csc x \cot x \, dx = \csc x + c$ OR $\int \csc x \cot x \, dx = -\csc x + c$

Section 2: Indefinite Integrals

$$\begin{array}{ccc} & f' & \\ \sin x & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & \cos x \\ & \int f dx & \end{array}$$

$$\begin{array}{ccc} & f' & \\ \cos x & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & -\sin x \\ & \int f dx & \\ \int \sin x dx = -\cos x + c & & \end{array}$$

$$\begin{array}{ccc} & f' & \\ \sec x & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & \sec x \tan x \\ & \int f dx & \end{array}$$

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$$\begin{array}{ccc} & f' & \\ \cot x & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & -\csc^2 x \\ & \int f dx & \\ \int \csc^2 x dx = -\cot x + c & & \end{array}$$

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Example

Evaluate the integral $\int (\cos x + \sec x \tan x) dx$

Section 2: Indefinite Integrals

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Example

Evaluate the integral $\int (\cos x + \sec x \tan x) dx$

Solution:

$$\int (\cos x + \sec x \tan x) dx = \int \cos x dx + \int \sec x \tan x dx = \sin x + \sec x + c$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

1 $\int \left(\frac{1}{\cos^2 x} - \sin x \right) dx$

2 $\int \sec x (\sec x + \tan x) dx$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$1 \int \left(\frac{1}{\cos^2 x} - \sin x \right) dx$$

$$2 \int \sec x (\sec x + \tan x) dx$$

Solution:

$$1 \int \left(\frac{1}{\cos^2 x} - \sin x \right) dx = \int \sec^2 x dx - \int \sin x dx = \tan x + \cos x + c .$$

Remember. $\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x}$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$① \int \left(\frac{1}{\cos^2 x} - \sin x \right) dx$$

$$② \int \sec x (\sec x + \tan x) dx$$

Solution:

$$① \int \left(\frac{1}{\cos^2 x} - \sin x \right) dx = \int \sec^2 x dx - \int \sin x dx = \tan x + \cos x + c .$$

Remember. $\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x}$

$$② \int \sec x (\sec x + \tan x) dx = \int \sec^2 x dx + \int \sec x \tan x dx = \tan x + \sec x + c .$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$① \int (4x + 3) dx$$

$$② \int (2 \sin x + 3 \cos x) dx$$

$$③ \int (\sqrt{x} + \sec^2 x) dx$$

$$④ \int \frac{d}{dx}(\sin x) dx$$

$$⑤ \frac{d}{dx} \int \sqrt{x+1} dx$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$① \int (4x + 3) dx$$

$$② \int (2 \sin x + 3 \cos x) dx$$

$$③ \int (\sqrt{x} + \sec^2 x) dx$$

$$④ \int \frac{d}{dx}(\sin x) dx$$

$$⑤ \frac{d}{dx} \int \sqrt{x+1} dx$$

Solution:

$$① \int (4x + 3) dx = 4 \int x dx + \int 3 dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx. \quad \text{and} \quad \int kf(x) dx = k \int f(x) dx$$

Section 2: Indefinite Integrals

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$$① \int (4x + 3) dx = 4 \int x dx + \int 3 dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx. \quad \text{and} \quad \int kf(x) dx = k \int f(x) dx$$

$$② \int (2 \sin x + 3 \cos x) dx = 2 \int \sin x dx + 3 \int \cos x dx = -2 \cos x + 3 \sin x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx. \quad \text{and} \quad \int kf(x) dx = k \int f(x) dx$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$\textcircled{1} \int (4x + 3) dx$$

$$\textcircled{2} \int (2 \sin x + 3 \cos x) dx$$

$$\textcircled{3} \int (\sqrt{x} + \sec^2 x) dx$$

$$\textcircled{4} \int \frac{d}{dx}(\sin x) dx$$

$$\textcircled{5} \frac{d}{dx} \int \sqrt{x+1} dx$$

Solution:

$$\textcircled{1} \int (4x + 3) dx = 4 \int x dx + \int 3 dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx. \quad \text{and} \quad \int kf(x) dx = k \int f(x) dx$$

$$\textcircled{2} \int (2 \sin x + 3 \cos x) dx = 2 \int \sin x dx + 3 \int \cos x dx = -2 \cos x + 3 \sin x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx. \quad \text{and} \quad \int kf(x) dx = k \int f(x) dx$$

$$\textcircled{3} \int (\sqrt{x} + \sec^2 x) dx = \int x^{\frac{1}{2}} dx + \int \sec^2 x dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tan x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan x + c.$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$\textcircled{1} \int (4x + 3) dx$$

$$\textcircled{2} \int (2 \sin x + 3 \cos x) dx$$

$$\textcircled{3} \int (\sqrt{x} + \sec^2 x) dx$$

$$\textcircled{4} \int \frac{d}{dx}(\sin x) dx$$

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Solution:

$$\textcircled{1} \int (4x + 3) dx = 4 \int x dx + \int 3 dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

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$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$$

$$\textcircled{4} \int \frac{d}{dx}(\sin x) dx = \sin x + c.$$

$$\int \frac{d}{dx}(F(x)) dx = F(x) + c.$$

Section 2: Indefinite Integrals

Example

Evaluate the integral.

$$\textcircled{1} \int (4x + 3) dx$$

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$$\textcircled{4} \int \frac{d}{dx}(\sin x) dx$$

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Solution:

$$\textcircled{1} \int (4x + 3) dx = 4 \int x dx + \int 3 dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

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$$\int \frac{d}{dx}(F(x)) dx = F(x) + c.$$

$$\textcircled{5} \frac{d}{dx} \int \sqrt{x+1} dx = \sqrt{x+1}.$$

$$\frac{d}{dx} \int f(x) dx = f(x).$$

Integration By Substitution

Remember. Rule 1

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

Example. $\int x^3 dx = \frac{x^4}{4} + c$

Integration By Substitution

Remember. Rule 1

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

Example. $\int x^3 dx = \frac{x^4}{4} + c$

Question. Can we use **Rule 1** to evaluate $\int 2x(x^2 + 1)^3 dx$?

Integration By Substitution

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Question. Can we use **Rule 1** to evaluate $\int 2x (x^2 + 1)^3 dx$?

Theorem

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on an interval J that contains the range of the function g . If $\int f(x) dx = F(x) + c$, then

$$\int f(g(x)) g'(x) dx = F(g(x)) + c, \quad \forall x \in I.$$

Integration By Substitution

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$$\int f(g(x)) g'(x) dx = F(g(x)) + c, \quad \forall x \in I.$$

Example

Evaluate the integral $\int 2x (x^2 + 1)^3 dx$.

Integration By Substitution

Remember. Rule 1

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

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Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on an interval J that contains the range of the function g . If $\int f(x) dx = F(x) + c$, then

$$\int f(g(x)) g'(x) dx = F(g(x)) + c, \quad \forall x \in I.$$

Example

Evaluate the integral $\int 2x(x^2 + 1)^3 dx$.

Solution: Let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $(f \circ g)(x) = f(g(x)) = (x^2 + 1)^3$.

$$g(x) = x^2 + 1 \Rightarrow g'(x) = 2x$$

From the theorem, we have

$$\int \underbrace{2x}_{g'(x)} \underbrace{(x^2 + 1)^3}_{f(g(x))} dx = \frac{(x^2 + 1)^4}{4} + c.$$

Integration By Substitution

Note. We can end with the same solution by using the five steps of the substitution method given below.

■ **Steps of the integration by substitution:**

Step 1: Choose a new variable u .

Step 2: Determine the value of du .

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral in terms of u .

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x .

Exercise. Evaluate the integral $\int u^3 du$

Solution. $\int u^3 du = \frac{u^4}{4} + c$

Example

Evaluate the integral $\int 2x(x^2 + 1)^3 dx$.

Solution: Let

$$u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow \frac{du}{2x} = dx$$

. By substituting that into the original integral, we have

$$\int 2x u^3 \frac{du}{2x} = \int u^3 du = \frac{u^4}{4} + c = \underbrace{\frac{(x^2 + 1)^4}{4}}_{\text{Returning the evaluation to } x} + c$$

Returning the evaluation to x

Integration By Substitution

Exercise. Evaluate the integral $\int \sqrt{x} \, dx$

Integration By Substitution

Exercise. Evaluate the integral $\int \sqrt{x} \, dx$

Solution. $\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2x^{\frac{3}{2}}}{3} + c$

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5} \, dx$

Integration By Substitution

Exercise. Evaluate the integral $\int \sqrt{x} \, dx$

Solution. $\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2x^{\frac{3}{2}}}{3} + c$

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5} \, dx$

Solution: $\int x^2 \sqrt{2x^3 - 5} \, dx = \int x^2 (2x^3 - 5)^{\frac{1}{2}} \, dx$

Integration By Substitution

Exercise. Evaluate the integral $\int \sqrt{x} \, dx$

Solution. $\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{x^{\frac{3}{2}}}{3/2} + c = \frac{2x^{\frac{3}{2}}}{3} + c$

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5} \, dx$

Solution: $\int x^2 \sqrt{2x^3 - 5} \, dx = \int x^2 (2x^3 - 5)^{\frac{1}{2}} \, dx$

Let $f(x) = x^{\frac{1}{2}}$ and $g(x) = 2x^3 - 5$, then $(f \circ g)(x) = f(g(x)) = (2x^3 - 5)^{\frac{1}{2}}$.

$$g(x) = 2x^3 - 5 \Rightarrow g'(x) = 6x^2$$

From the theorem $\int f(g(x))g'(x) \, dx = F(g(x)) + c$, we have

$$\frac{1}{6} \int \underbrace{6x^2}_{g'(x)} \underbrace{(2x^3 - 5)^{\frac{1}{2}}}_{f(g(x))} \, dx = \frac{1}{6} \frac{(2x^3 - 5)^{\frac{3}{2}}}{3/2} + c = \frac{1}{6} \frac{2}{3} (2x^3 - 5)^{\frac{3}{2}} + c = \frac{(2x^3 - 5)^{\frac{3}{2}}}{9} + c.$$

Integration By Substitution

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5} dx$

Integration By Substitution

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5} dx$

Solution:

$$\int x^2 \sqrt{2x^3 - 5} dx = \int x^2 (2x^3 - 5)^{\frac{1}{2}} dx$$

Let

$$u = 2x^3 - 5 \Rightarrow du = 6x^2 dx \Rightarrow \frac{du}{6x^2} = dx$$

By substitution, we have

$$\int x^2 u^{\frac{1}{2}} \frac{du}{6x^2} = \frac{1}{6} \int u^{\frac{1}{2}} du = \frac{1}{6} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{6} \frac{2}{3} u^{\frac{3}{2}} + c = \frac{u^{\frac{3}{2}}}{9} + c = \underbrace{\frac{(2x^3 - 5)^{\frac{3}{2}}}{9}}_{\text{Returning the evaluation to } x} + c$$

Integration By Substitution

Exercise. Evaluate the integral $\int \sec^2 x \, dx$

Solution. $\int \sec^2 x \, dx = \tan x + c$

Integration By Substitution

Exercise. Evaluate the integral $\int \sec^2 x \, dx$

Solution. $\int \sec^2 x \, dx = \tan x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$

Integration By Substitution

Exercise. Evaluate the integral $\int \sec^2 x \, dx$

Solution. $\int \sec^2 x \, dx = \tan x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$

Solution: Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = f(g(x)) = \sec^2 \sqrt{x}$.

$$g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

From the theorem $\int f(g(x))g'(x) \, dx = F(g(x)) + c$, we have

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \sec^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

Integration By Substitution

Exercise. Evaluate the integral $\int \sec^2 x \, dx$

Solution. $\int \sec^2 x \, dx = \tan x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$

Solution: Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = f(g(x)) = \sec^2 \sqrt{x}$.

$$g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

From the theorem $\int f(g(x))g'(x) \, dx = F(g(x)) + c$, we have

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \sec^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$.

Solution: Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2\sqrt{x} \, du = dx$. By substitution, we obtain

$$\int \frac{\sec^2 u}{\sqrt{x}} \cdot 2\sqrt{x} \, du = 2 \int \sec^2 u \, du = 2 \tan u + c = 2 \tan \sqrt{x} + c$$

Integration By Substitution

Example

Evaluate the integral

1 $\int \sqrt{2x - 5} \, dx$

2 $\int \cos(3x + 4) \, dx$

Integration By Substitution

Example

Evaluate the integral

1 $\int \sqrt{2x - 5} \, dx$

2 $\int \cos(3x + 4) \, dx$

Solution:

1 $\int \sqrt{2x - 5} \, dx = \int (2x - 5)^{\frac{1}{2}} \, dx$

Integration By Substitution

Example

Evaluate the integral

① $\int \sqrt{2x - 5} \, dx$

② $\int \cos(3x + 4) \, dx$

Solution:

① $\int \sqrt{2x - 5} \, dx = \int (2x - 5)^{\frac{1}{2}} \, dx$

Let $u = 2x - 5 \Rightarrow du = 2 \, dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} \, du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} \frac{u^{\frac{3}{2}}}{2} + c = \frac{(2x - 5)^{\frac{3}{2}}}{3} + c$$

Integration By Substitution

Example

Evaluate the integral

① $\int \sqrt{2x - 5} \, dx$

② $\int \cos(3x + 4) \, dx$

Solution:

① $\int \sqrt{2x - 5} \, dx = \int (2x - 5)^{\frac{1}{2}} \, dx$

Let $u = 2x - 5 \Rightarrow du = 2 \, dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} \, du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{2} \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x - 5)^{\frac{3}{2}}}{3} + c$$

OR

$$\int \sqrt{2x - 5} \, dx = \int (2x - 5)^{\frac{1}{2}} \, dx = \frac{1}{2} \int 2 (2x - 5)^{\frac{1}{2}} \, dx = \frac{1}{2} \frac{(2x - 5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x - 5)^{\frac{3}{2}}}{3} + c$$

Integration By Substitution

Example

Evaluate the integral

① $\int \sqrt{2x-5} \, dx$

② $\int \cos(3x+4) \, dx$

Solution:

① $\int \sqrt{2x-5} \, dx = \int (2x-5)^{\frac{1}{2}} \, dx$

Let $u = 2x - 5 \Rightarrow du = 2 \, dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} \, du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{2} \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

OR

$$\int \sqrt{2x-5} \, dx = \int (2x-5)^{\frac{1}{2}} \, dx = \frac{1}{2} \int 2 (2x-5)^{\frac{1}{2}} \, dx = \frac{1}{2} \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

② $\int \cos(3x+4) \, dx = \frac{1}{3} \int 3 \cos(3x+4) \, dx = \frac{1}{3} \sin(3x+4) + c$

Integration By Substitution

Example

Evaluate the integral

1 $\int 5x(x^2 + 3)^7 dx$

2 $\int \sec^2(4x) dx$

Integration By Substitution

Example

Evaluate the integral

1 $\int 5x(x^2 + 3)^7 dx$

2 $\int \sec^2(4x) dx$

Solution:

1 $5 \int x(x^2 + 3)^7 dx = \frac{5}{2} \int 2x(x^2 + 3)^7 dx = \frac{5}{2} \frac{(x^2 + 3)^8}{8} + c = 5 \frac{(x^2 + 3)^8}{16} + c$

Integration By Substitution

Example

Evaluate the integral

$$1 \quad \int 5x(x^2 + 3)^7 dx$$

$$2 \quad \int \sec^2(4x) dx$$

Solution:

$$1 \quad 5 \int x(x^2 + 3)^7 dx = \frac{5}{2} \int 2x(x^2 + 3)^7 dx = \frac{5}{2} \frac{(x^2 + 3)^8}{8} + c = 5 \frac{(x^2 + 3)^8}{16} + c$$

$$2 \quad \frac{1}{4} \int 4 \sec^2(4x) dx = \frac{1}{4} \tan(4x) + c$$

Integration By Substitution

Example

Evaluate the integral

$$1 \quad \int 5x(x^2 + 3)^7 dx$$

$$2 \quad \int \sec^2(4x) dx$$

Solution:

$$1 \quad 5 \int x(x^2 + 3)^7 dx = \frac{5}{2} \int 2x(x^2 + 3)^7 dx = \frac{5}{2} \frac{(x^2 + 3)^8}{8} + c = 5 \frac{(x^2 + 3)^8}{16} + c$$

$$2 \quad \frac{1}{4} \int 4 \sec^2(4x) dx = \frac{1}{4} \tan(4x) + c$$