Ch.10: Rotation of a Rigid Object About a Fixed Axis

Physics 103: Classical Mechanics

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Version 1

Outline

1. **Rigid Body**
2. Angular Position, Velocity, and Acceleration
3. Rotational Motion with Constant Angular Acceleration
4. Angular and Linear Quantities
5. Rotational Kinetic Energy
6. Calculation of Moments of Inertia
7. Torque
8. Relationship Between Torque and Angular Acceleration
9. Work, Power, and Energy in Rotational Motion

1.1 Definition

- A rigid body is an object that does not deform or change its shape during motion. The distance between any two points in a rigid body remains constant regardless of the motion of the body.

1.2 Point Objects to Rigid Objects

- So far we have been simplify objects as point masses to make calculations easier.
- However, when dealing with rigid bodies, we need to consider their size and shape, as well as how they rotate about an axis.
- Therefore, we can no longer treat them as point masses, and we need to use rotational dynamics to describe their motion.

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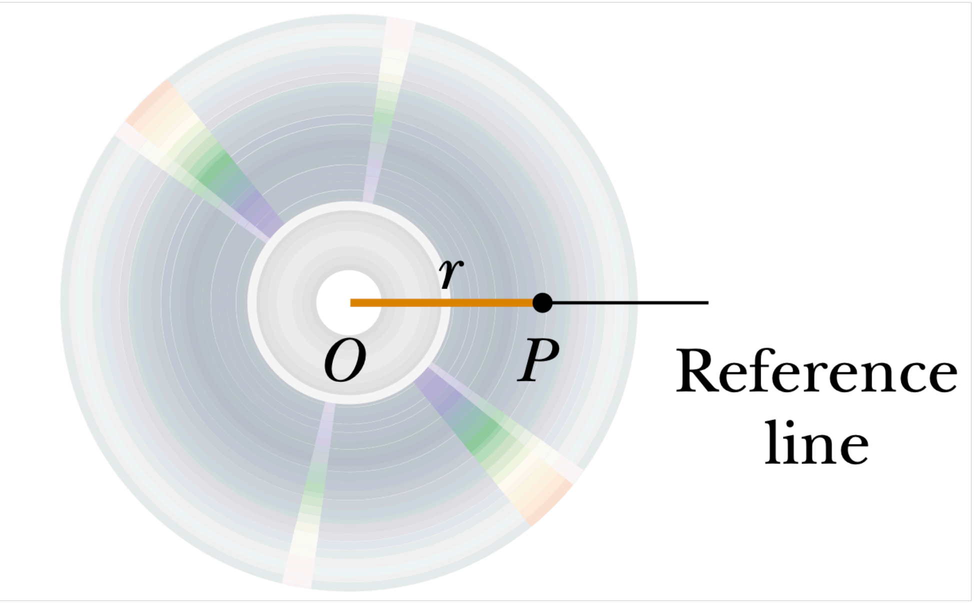
2.1 Angular Position

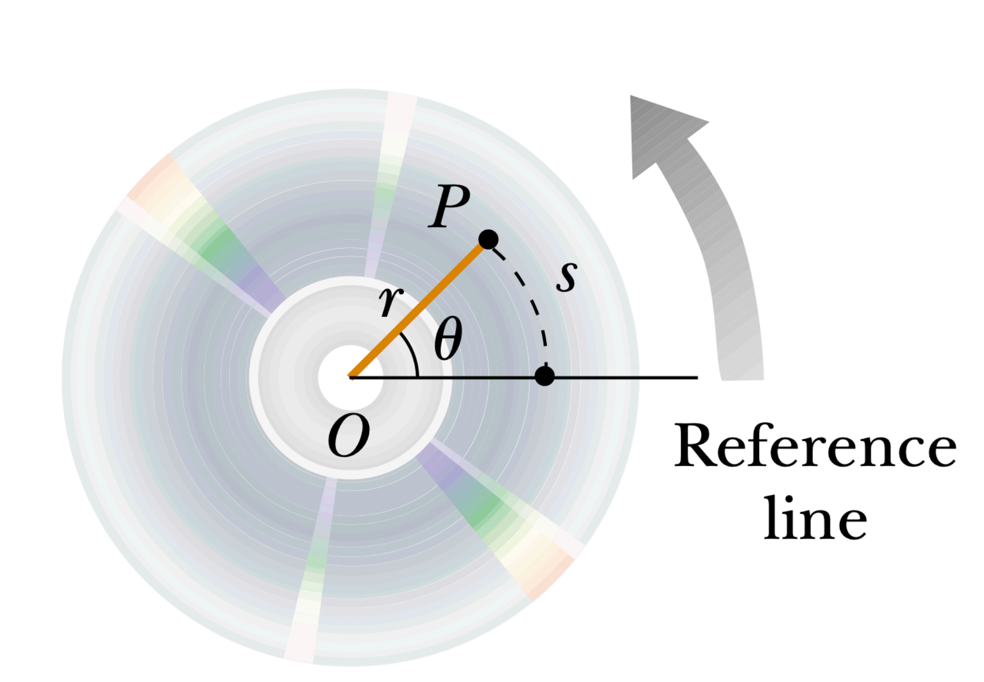
- The position of a point on a rigid body can be described using the angle θ and a radius r from a reference point.



2.1 Angular Position

- The arc length s that the point travels is proportional to the angle θ and the radius r by the equation:

$$s = r\,\theta$$



2.1 Angular Position

- Rearranging this equation gives us the angular position:

$$\theta = \frac{s}{r}$$

- Notice that theta has no unit, however, it is normally expressed in radians.
- 1 radians is equivalent to an arc length equal to the radius of the circle.
- Since the circumference of a circle is $2\pi r$, then one complete revolution is equal to $\theta = 2\pi$ radians.

2.1 Angular Position

- To convert from degrees to radians, use the relation:

$$\theta(\text{rad}) = \frac{\pi}{180°}\,\theta(\text{deg}) \approx \frac{3.14}{180}\,\theta(\text{deg})$$

To convert from revolutions to radians, use the relation:

$$\theta(\text{rad}) = 2\pi(\text{Number of rev})$$

To find the change in angular position, we use:

$$\Delta\theta = \theta_f - \theta_i$$

where θ_f and θ_i are the final and initial angular positions, respectively.

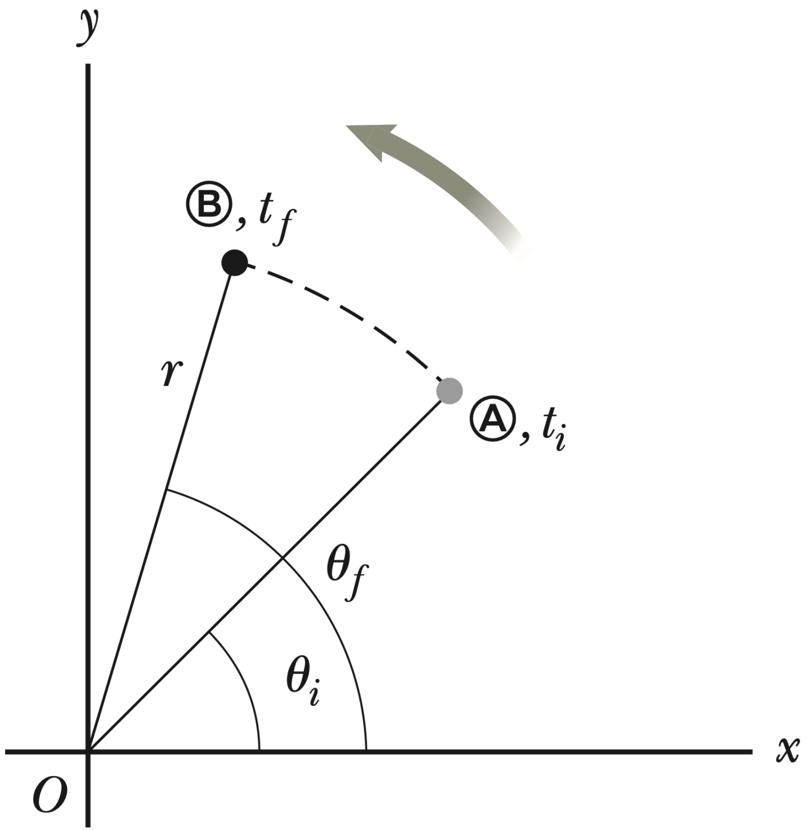
2.2 Angular speed

Average Angular Speed

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Speed

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$



2.3 Angular Acceleration

Average Angular Acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

2.4 Directions of Angular Velocity and Acceleration

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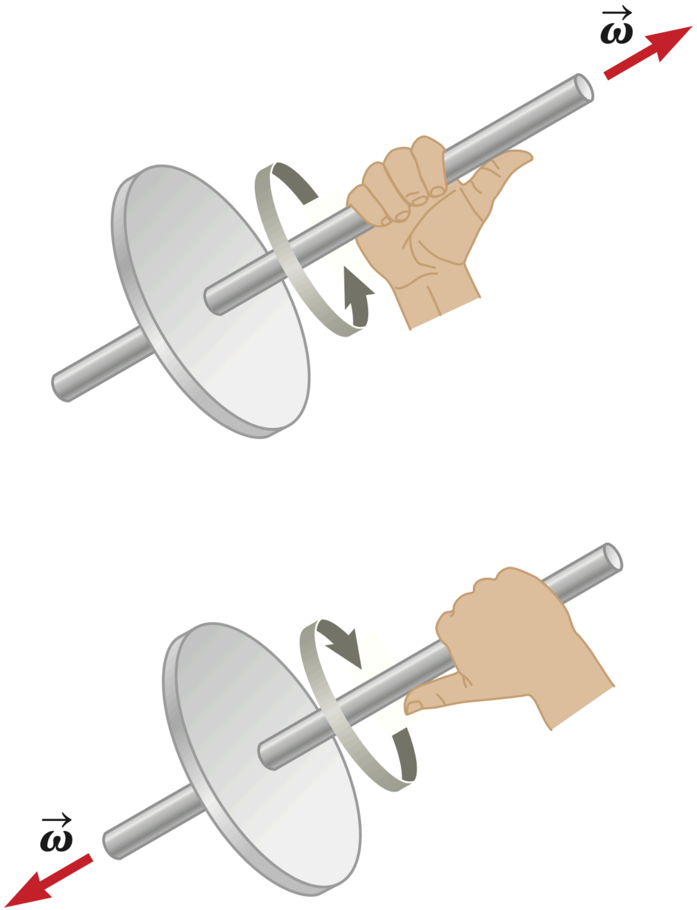
- The direction of $\vec{\omega}$ is given by the right-hand rule:
- If the fingers of the right hand curl in the direction of rotation, the thumb points in the direction of $\vec{\omega}$.
- Therefore, there are only two possible directions for $\vec{\omega}$: positive or negative along the axis of rotation.
- Similarly, $\vec{\alpha}$ also can point in either direction along the axis of rotation, depending on whether the angular speed is increasing or decreasing.



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3.1 Equations of Angular motion with constant acceleration

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

3.1 Equations of Angular motion with constant acceleration

Example 3.1

A wheel rotates with a constant angular acceleration of 3.5 rad/s^2.

(A) If the angular speed of the wheel is 2 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2 s?

$$\Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$$

$$= (2\ \text{rad/s})(2s) + \frac{1}{2}(3.5\ \text{rad/s}^2)(2s)^2$$

$$= 11\ \text{rad} = 11\ \text{rad}\ \frac{57.3°}{1\ \text{rad}} \approx 630°$$

3.1 Equations of Angular motion with constant acceleration

(B) Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 11 \text{ rad} \frac{1 \text{ rev}}{2\pi \text{ rad}} \approx 1.75 \text{ rev}$$

(C) What is the angular speed of the wheel at t 2 s?

$$\omega_f = \omega_i + \alpha t = 2 \text{ rad/s} + (3.5 \text{ rad/s}^2)(2s) = 9 \text{ rad/s}$$

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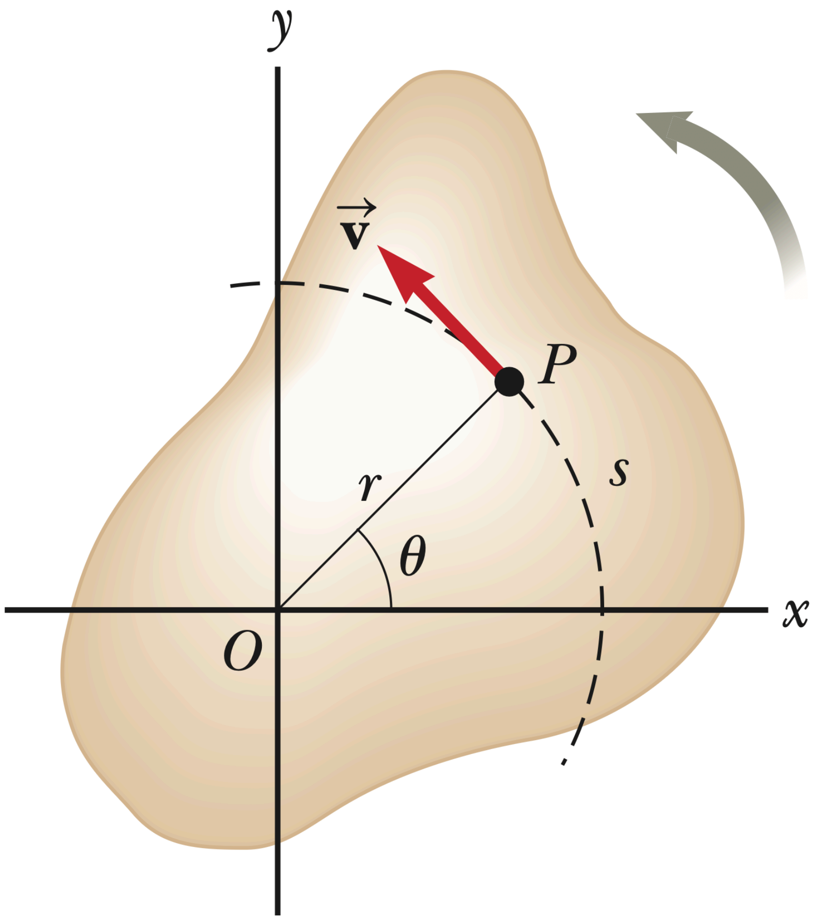
4.1 Velocity

- The linear velocity v of a point on a rotating rigid body is related to its angular velocity ω by the equation:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Therefore

$$\boxed{v = r\,\omega}$$



4.2 Acceleration

- The tangential acceleration a_t of a point on a rotating rigid body is related to its angular acceleration α by the equation:

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = r\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Therefore,

$$a_t = r\alpha$$

- Also, we can rewrite the centripetal acceleration a_c, to:

$$a_c = \frac{v^2}{r} = r\omega^2$$

4.2 Acceleration

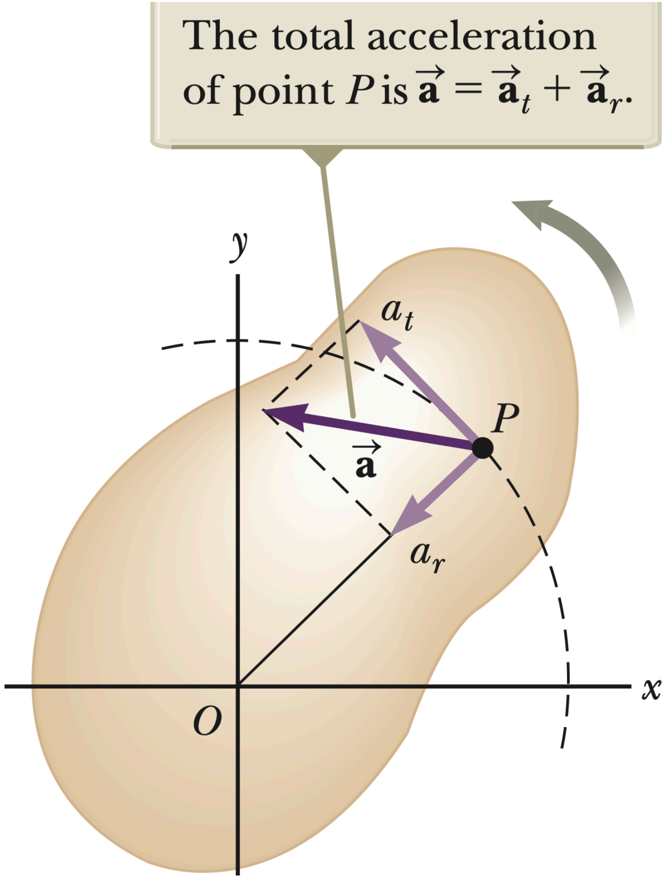
Total acceleration

The total acceleration a_{tot} of a point on a rotating rigid body is the vector sum of its tangential acceleration a_t and centripetal acceleration a_c. Its magnitude is given by:

$$a_{\text{tot}} = \sqrt{a_t^2 + a_c^2}$$

$$= \sqrt{(r\alpha)^2 + (r\omega^2)^2}$$

$$a_{\text{tot}} = r\sqrt{\alpha^2 + \omega^4}$$

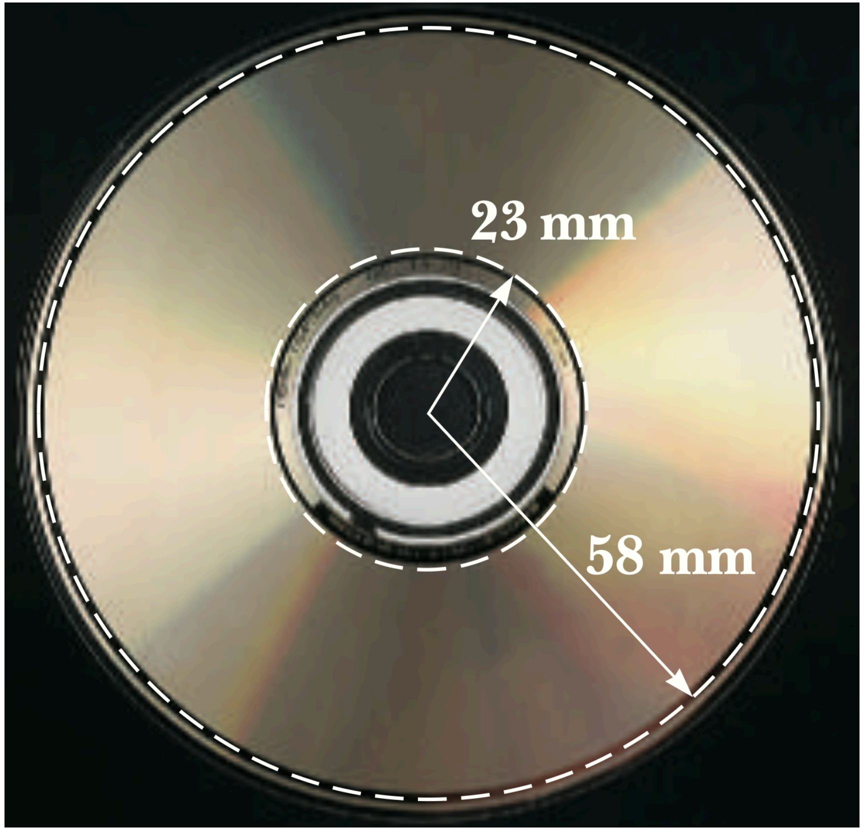


4.2 Acceleration

Example 4.2

In a typical compact disc player, the constant speed of the surface at the point of the laser-lens system is 1.3 m/s.

(A) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track (r = 23 mm) and the outermost final track (r = 58 mm).



4.2 Acceleration

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Solution 4.2

- For the innermost track:

$$\omega_i = \frac{v}{r_i} = \frac{1.3 \text{ m/s}}{23 \times 10^{-3} \text{ m}}$$
$$= 57 \text{ rad/s}$$
$$= 57 \text{ rad/s} \left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right)$$
$$= 540 \text{ rev/min}$$

- For the outermost track:

$$\omega_f = \frac{v}{r_f} = \frac{1.3 \text{ m/s}}{58 \times 10^{-3} \text{ m}}$$
$$= 22 \text{ rad/s}$$
$$= 210 \text{ rev/min}$$

4.2 Acceleration

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(B) The maximum playing time of a standard CD is 74 min and 33 s. How many revolutions does the disc make during that time?

$$\begin{aligned}\Delta\theta &= \frac{1}{2}(\omega_i + \omega_f)t \\ &= \frac{1}{2}(57 \text{ rad/s} + 22 \text{ rad/s})(74 \text{ min} * 60 \text{ s/min} + 33 \text{ s}) \\ &= 1.8 \times 10^5 \text{ rad} \\ &= 2.8 \times 10^4 \text{ rev}\end{aligned}$$

4.2 Acceleration

(C) What total length of track moves past the objective lens during this time?

$$x = vt = (1.3 \text{ m/s})(74 \text{ min} * 60 \text{ s/min} + 33 \text{ s})$$
$$= 5.8 \times 10^3 \text{ m} = 5.8 \text{ km}$$

4.2 Acceleration

(D) What is the angular acceleration of the CD over the 4 473s time interval? Assume that α is constant.

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

$$= \frac{22 \text{ rad/s} - 57 \text{ rad/s}}{4473 \text{ s}}$$

$$= -7.8 \times 10^{-3} \text{ rad/s}^2$$

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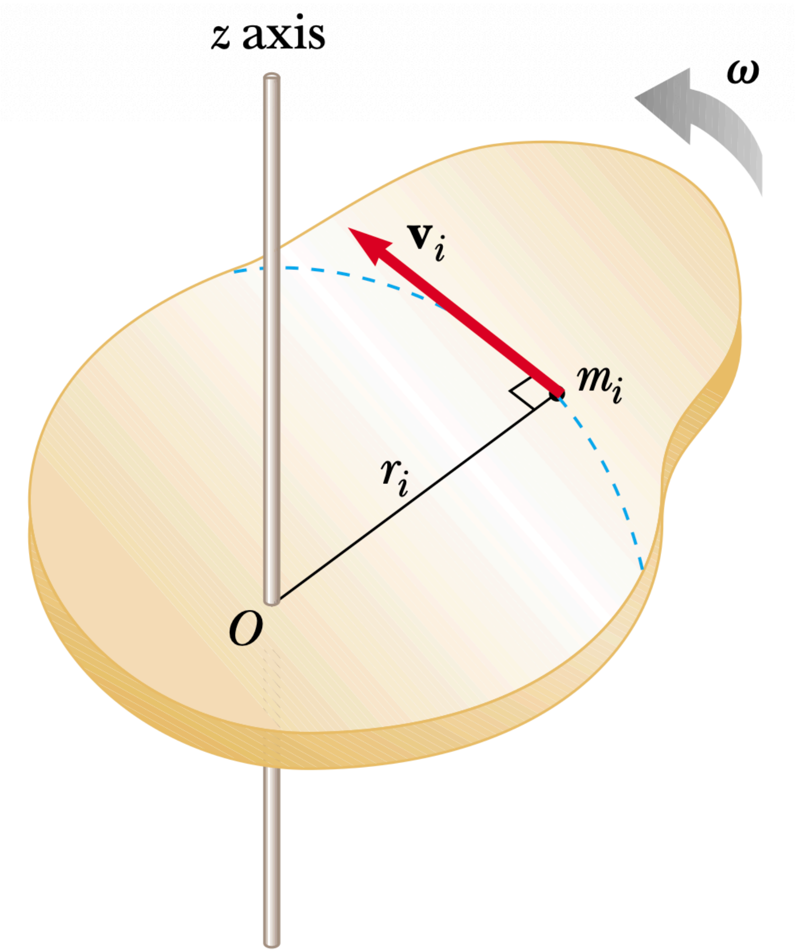
5.1 Definition and Derivation



- A rigid object rotating around an axis with angular velocity ω, the kinetic energy K_i of a tiny mass m_i at a distance r_i from the axis of rotation is given by:

$$\begin{aligned} K_i &= \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} m_i (r_i \omega)^2 \\ &= \frac{1}{2} m_i r_i^2 \omega^2 \\ &= \frac{1}{2} (m_i r_i^2) \omega^2 \end{aligned}$$

5.1 Definition and Derivation

- The total rotational kinetic energy K_R of the rigid object is the sum of the kinetic energies of all its tiny masses:

$$K_R = \sum_i K_i$$
$$= \sum_i \frac{1}{2}(m_i r_i^2)\omega^2$$
$$= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$K_R = \frac{1}{2} I \omega^2$$

where

$$I = \sum_i m_i r_i^2$$

Is the **moment of inertia** of the rigid object about the axis of rotation.

5.1 Definition and Derivation

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Example 5.3

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is $m = 2.66 \times 10^{-26}$ kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m.

(A) Calculate the moment of inertia of the molecule about the z axis.

5.1 Definition and Derivation

Solution 5.3

$$I = \sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2$$

$$= 2m\left(\frac{d}{2}\right)^2 = \frac{1}{2}md^2$$

$$= \frac{1}{2}(2.66 \times 10^{-26}\text{kg})\left(1.21 \times 10^{-10}\ \text{m}\right)^2$$

$$= 1.95 \times 10^{-46}\ \text{kg}\ \ \text{m}^2.$$

5.1 Definition and Derivation

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(B) If the angular speed of the molecule about the z-axis is $\omega = 4.6 \times 10^{12}$ rad/s, what is its rotational kinetic energy?

$$K_R = \frac{1}{2} I \omega^2$$
$$= \frac{1}{2}(1.95 \times 10^{-46}\ \text{kg m}^2)(4.6 \times 10^{12}\ \text{rad/s})^2$$
$$= 2.1 \times 10^{-21}\ \text{J}.$$

5.1 Definition and Derivation

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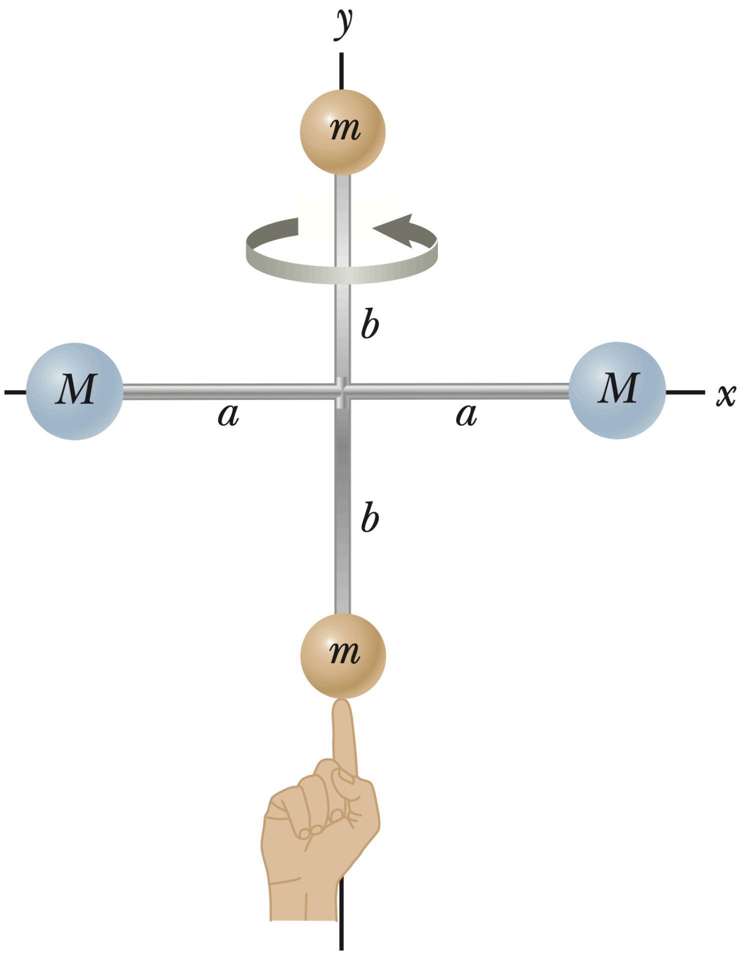
Example 5.4

Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane. Assume that the radii of the spheres are small compared with the dimensions of the rods.

(A) If the system rotates about the y axis with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.



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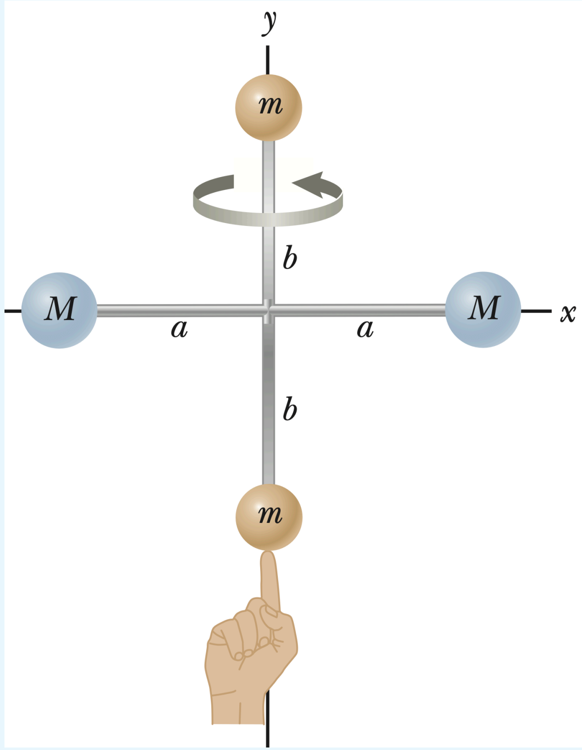
Solution 5.4



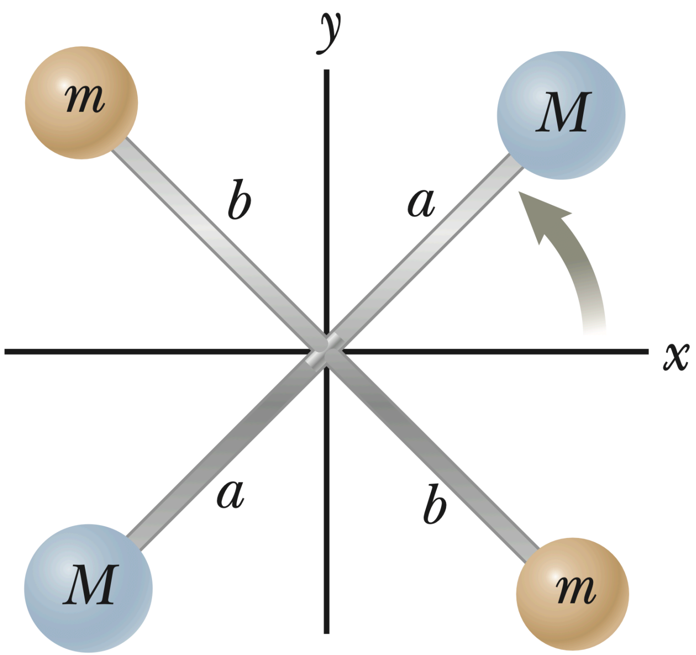
$$I_y = m(0)^2 + Ma^2 + m(0)^2 + Ma^2$$
$$= 2Ma^2$$

Therefore, the rotational kinetic energy about the y-axis is:

$$K_R = \frac{1}{2} I_y \omega^2$$
$$= \frac{1}{2}(2Ma^2)\omega^2$$
$$= Ma^2\omega^2$$

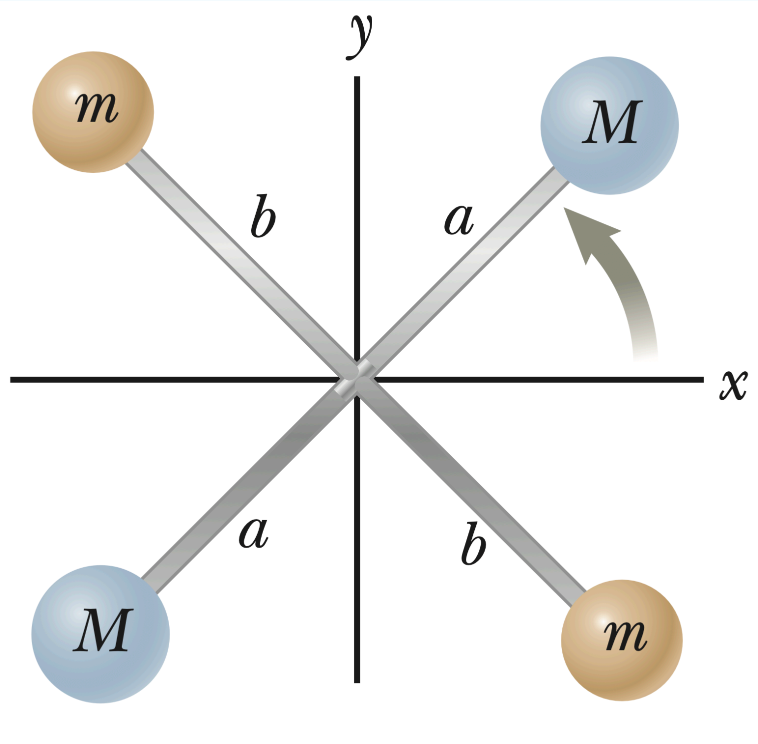
5.1 Definition and Derivation

(B) Suppose the system rotates in the xy plane about an axis (the z axis) through O. Calculate the moment of inertia and rotational kinetic energy about this axis.



5.1 Definition and Derivation

Solution 5.4



$$
\begin{aligned}
I &= Ma^2 + mb^2 + Ma^2 + mb^2 \\
&= 2Ma^2 + 2mb^2
\end{aligned}
$$

$$
\begin{aligned}
K_R &= \frac{1}{2} I\omega^2 \\
&= \frac{1}{2}(2Ma^2 + 2mb^2)\omega^2 \\
&= (Ma^2 + mb^2)\omega^2
\end{aligned}
$$

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6.1 Integral Form of the Moment of Inertia

- For a continuous rigid body, the moment of inertia I about an axis of rotation is given by the integral:

$$I = \int r^2 \, \mathrm{d}m$$

where r is the perpendicular distance from the axis of rotation to the mass element $\mathrm{d}m$.

- To evaluate this integral, we typically express $\mathrm{d}m$ in terms of a volume element $\mathrm{d}V$ and the density $\rho = M/V$ of the material:

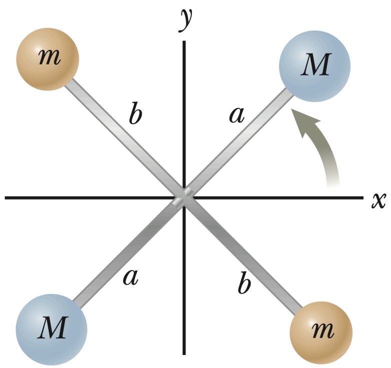
$$\mathrm{d}m = \rho \; \mathrm{d}V$$

$$I = \int \rho \, r^2 \, \mathrm{d}V$$

6.2 Four Types of densities or Mass distributions

1. **Point Masses**

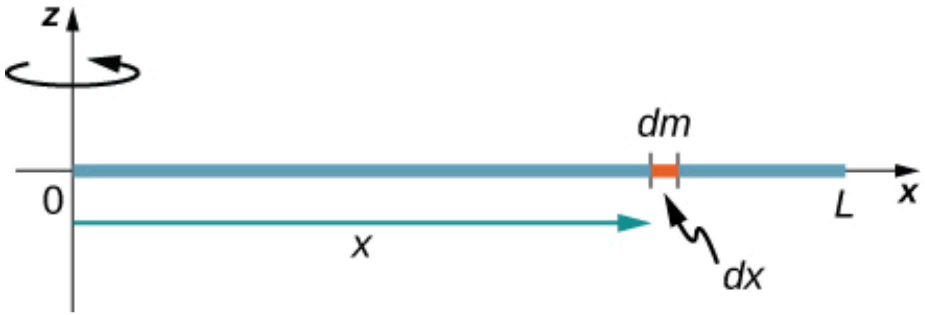
Discrete objects of individual point masses.



$$I = \sum_i m_i r_i^2$$

2. **Linear density**

For one-dimensional objects like rods.



$$\lambda = \frac{M}{L} = \rho A, \quad \text{A: cross-sectional area}$$

$$\boxed{dm = \lambda\, dx}$$

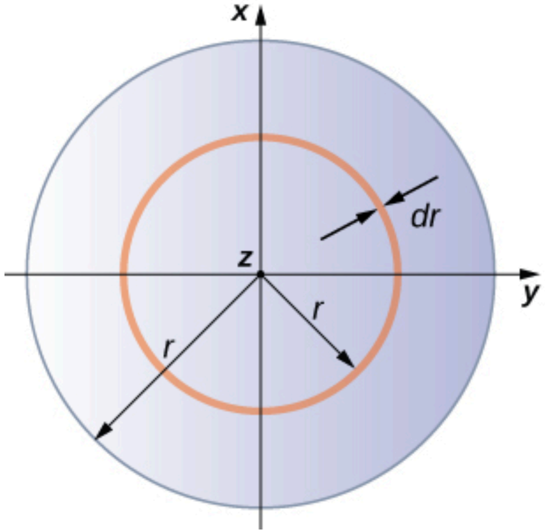
6.2 Four Types of densities or Mass distributions

3. **Surface density**

For 2D objects like plates.

$$\sigma = \frac{M}{A} = \rho t,$$

t: thickness

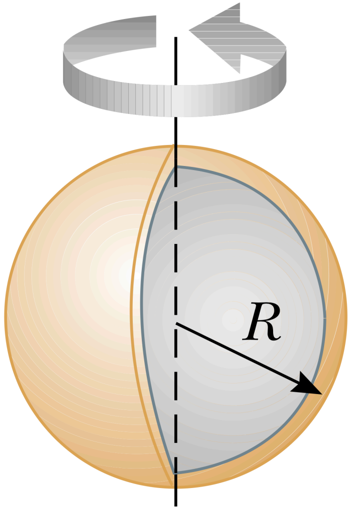


$$\mathrm{d}m = \sigma\,\mathrm{d}A$$

4. **Volume density**

For 3D objects like solid bodies.

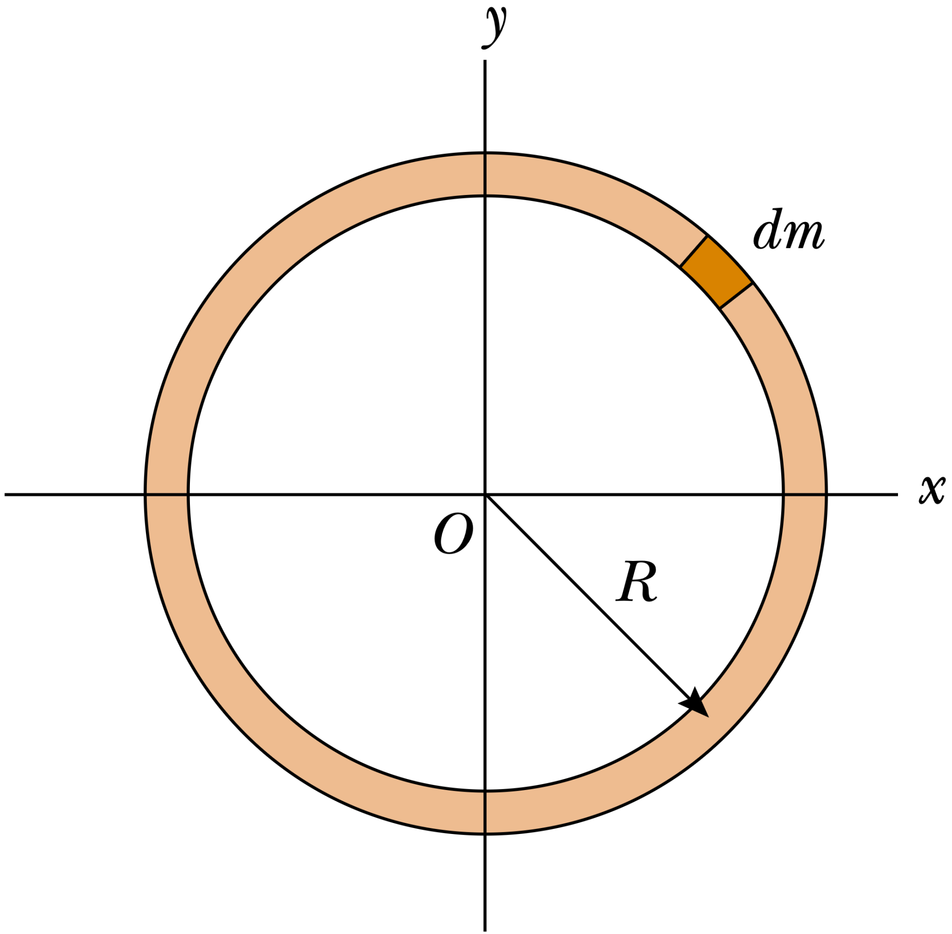
$$\rho = \frac{M}{V}$$



$$\mathrm{d}m = \rho\,\mathrm{d}V$$

Example 6.5

Find the moment of inertia of a uniform thin hoop (طوق) of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



6.3 Examples

Solution 6.5

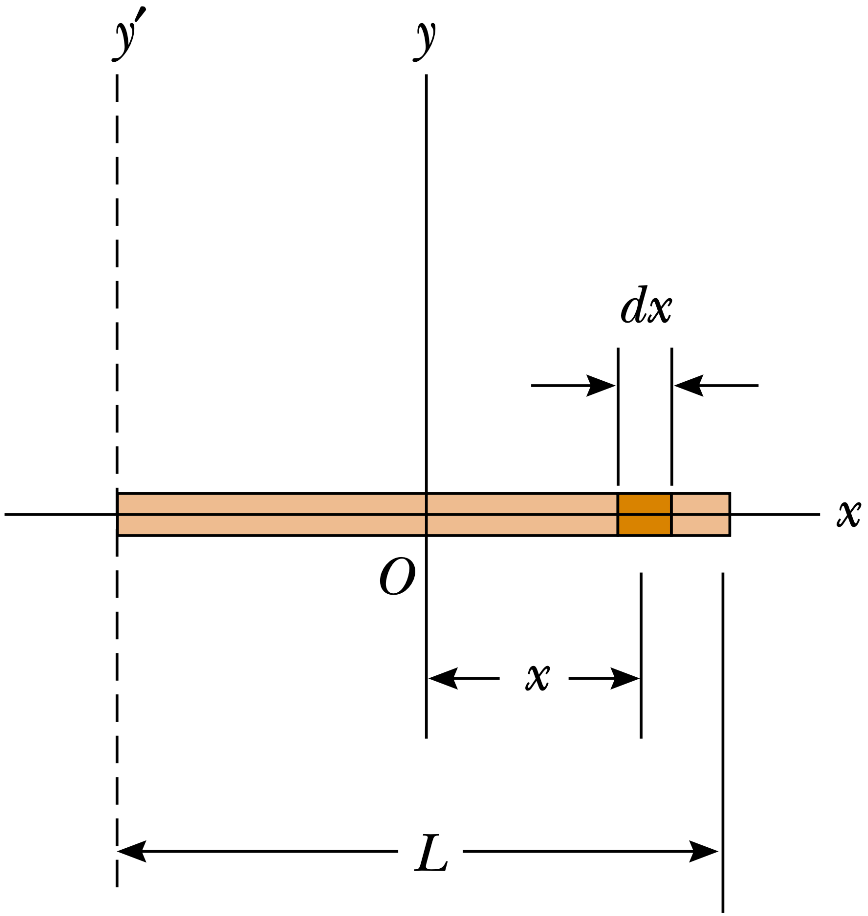
- We assume that the hoop is very thin, such that all its mass is concentrated at a distance R from the axis of rotation.
- Therefore, the moment of inertia can be calculated using:

$$I_z = \int r^2 \, \mathrm{d}m = R^2 \int \mathrm{d}m = R^2 M.$$

6.3 Examples

Example 6.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y axis) and passing through its center of mass.



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Solution 6.6

$$\mathrm{d}m = \lambda \, \mathrm{d}x = \left(\frac{M}{L}\right) \mathrm{d}x$$

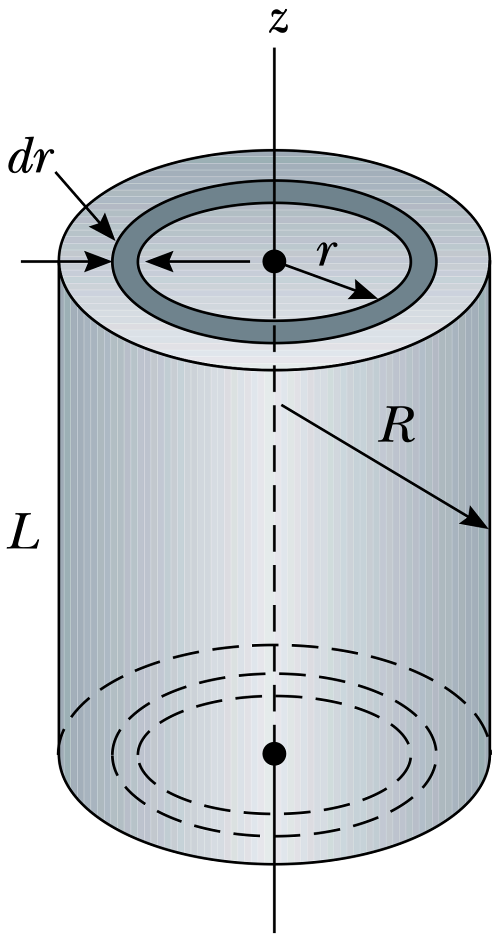
Therefore,

$$I_y = \int r^2 \, \mathrm{d}m = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L}\right) \mathrm{d}x = \left(\frac{M}{L}\right) \left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

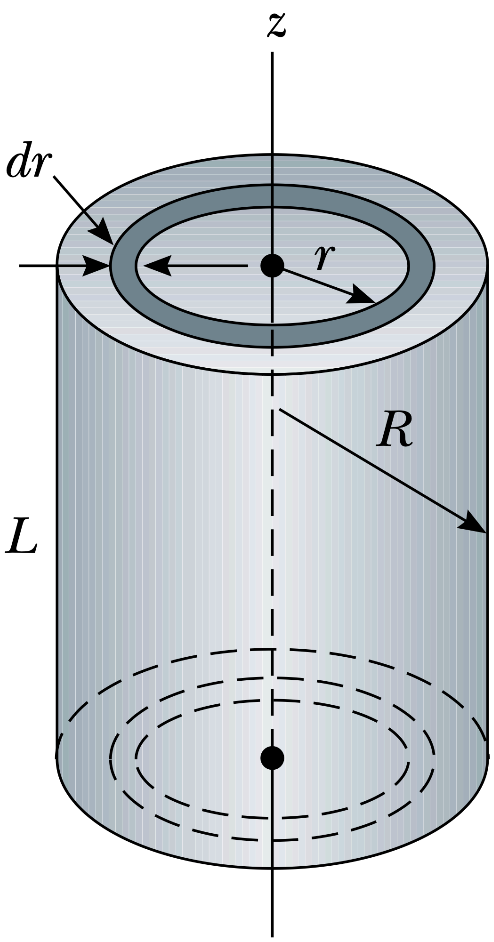
$$= \left(\frac{M}{L}\right) \left(\frac{L^3}{24} + \frac{L^3}{24}\right) = \left(\frac{1}{12}\right) ML^2.$$

Example 6.7

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the z axis).



6.3 Examples



Solution 6.7

$$I_z = \int r^2 \, \mathrm{d}m = \int r^2 \rho \, \mathrm{d}V$$

It is convenient to divide the cylinder into thin cylindrical shells of radius r, thickness $\mathrm{d}r$ and length L. Therefore, the volume element of each shell is given by:

$$\mathrm{d}V = L(2\pi r)\,\mathrm{d}r$$

and the mass element is:

$$\mathrm{d}m = \rho\,\mathrm{d}V = 2\pi\rho L r\,\mathrm{d}r$$

6.3 Examples

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Substituting into the integral for I_z, we get:

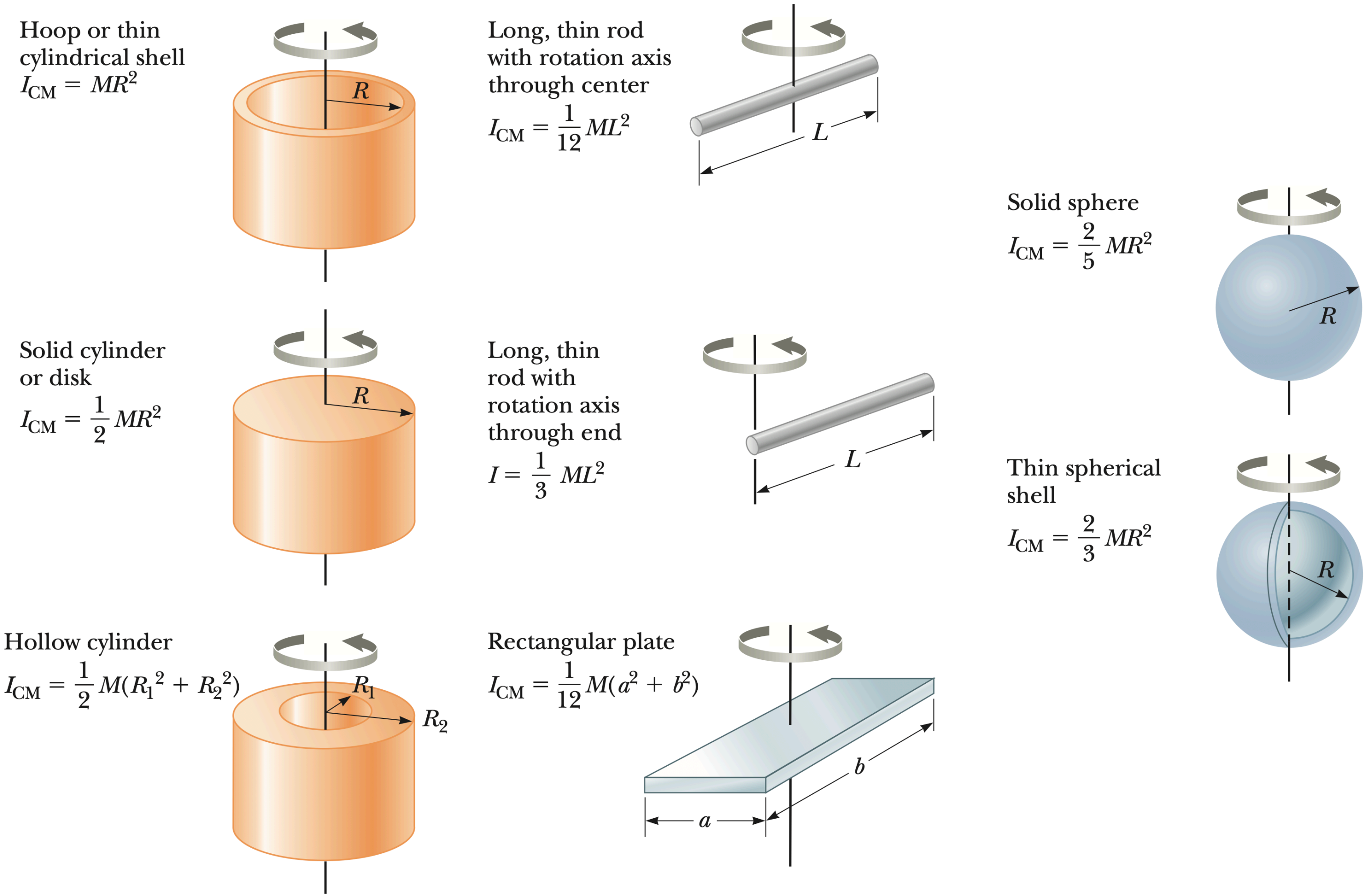
$$I_z = \int_0^R r^2(2\pi\rho L r)\,\mathrm{d}r = 2\pi\rho L \int_0^R r^3\,\mathrm{d}r$$

$$= 2\pi\rho L \left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}\pi\rho L R^4.$$

Substituting $\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$ into the expression for I_z, we obtain:

$$I_z = \frac{1}{2}\pi\left(\frac{M}{\pi R^2 L}\right) L R^4 = \left(\frac{1}{2}\right) M R^2$$

6.3 Examples

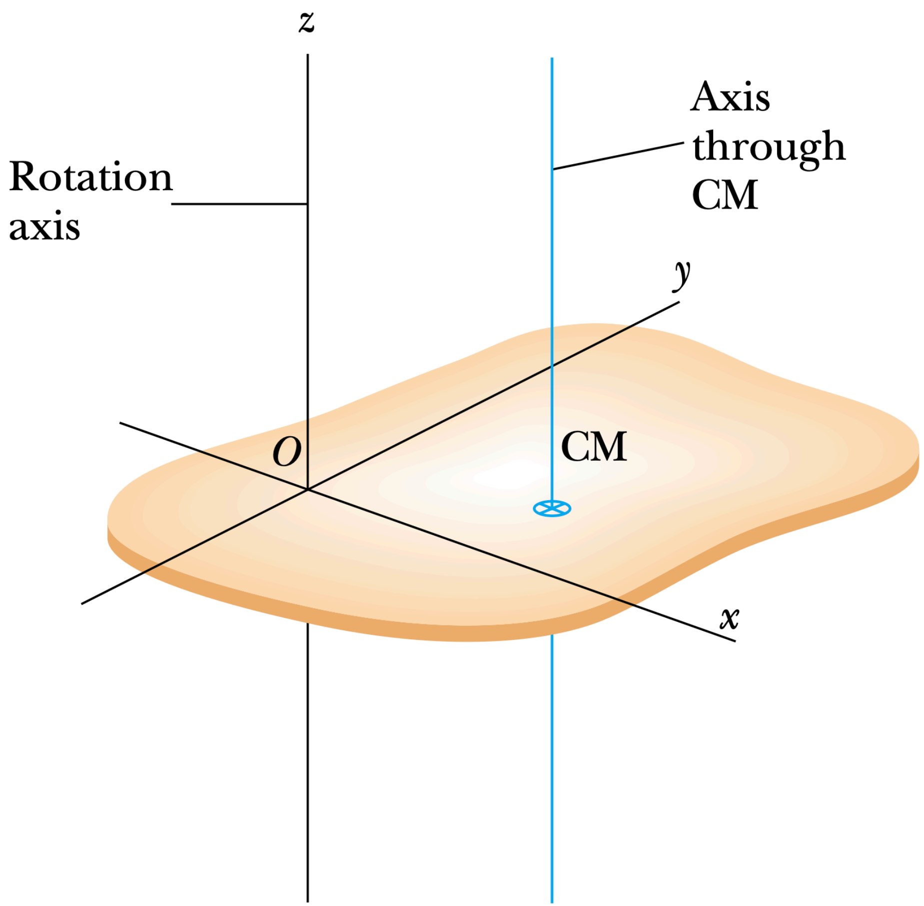


Hoop or thin cylindrical shell
$I_{CM} = MR^2$
R
Long, thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$
L
Solid sphere
$I_{CM} = \frac{2}{5}MR^2$
R
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$
R
Long, thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$
L
Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$
R
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$
R_1
R_2
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$
b
a

6.4 Center of Mass (CM)

- Each rigid body has a point called the center of mass (CM) where we can consider all the mass of the body to be concentrated for translational motion analysis to be simplified.
- For a symmetrical object with uniform density, the CM is located at its geometric center.
- When I is calculated about an axis passing through its CM, it is denoted as I_{CM}.
- Typically , calculating I_{CM} is easier than calculating I about any other axis.

6.5 Parallel-Axis Theorem



- The parallel-axis theorem states that the moment of inertia I of a rigid body about any axis parallel to and a distance D away from I_{CM} through its center of mass is given by:

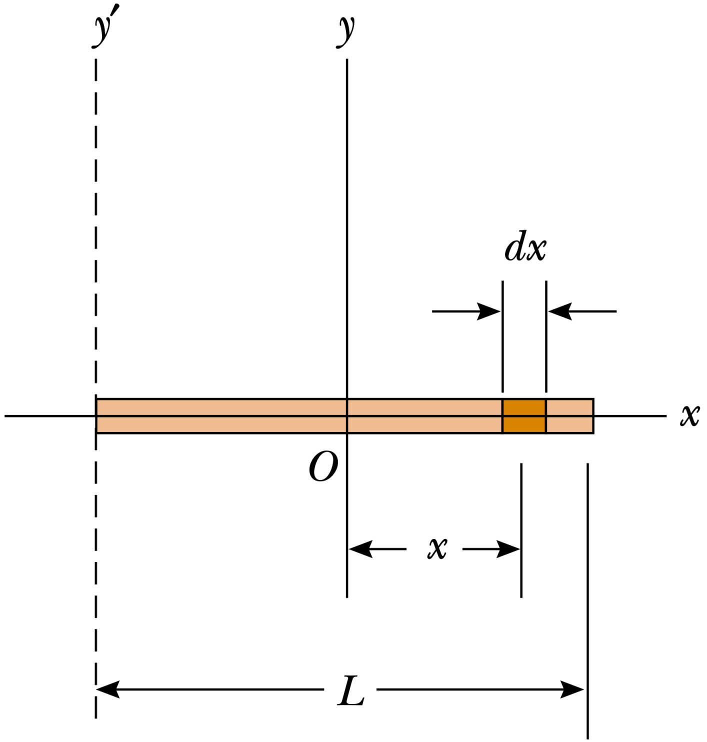
$$I = I_{\mathrm{cm}} + MD^2$$

where I_{cm} is the moment of inertia about the center of mass axis, and M is the total mass of the body.

6.6 Example

Example 6.8

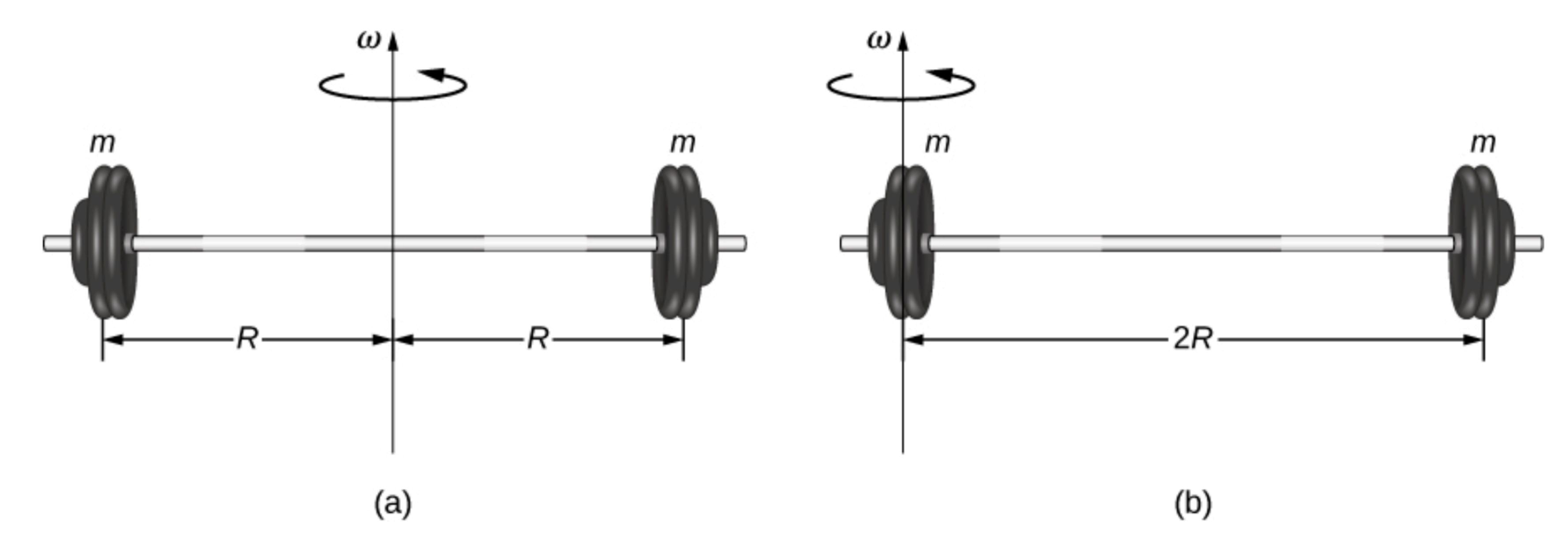
Consider once again the uniform rigid rod of mass M and length L shown in the Figure. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end (the y' axis).



$$I = I_{\text{CM}} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

6.6 Example

Quiz



Which one is easier to rotate?
The answer is (A) because the rotational axis passes through the center of mass.

1. Rigid Body

2. Angular Position, Velocity, and Acceleration

3. Rotational Motion with Constant Angular Acceleration

4. Angular and Linear Quantities

5. Rotational Kinetic Energy

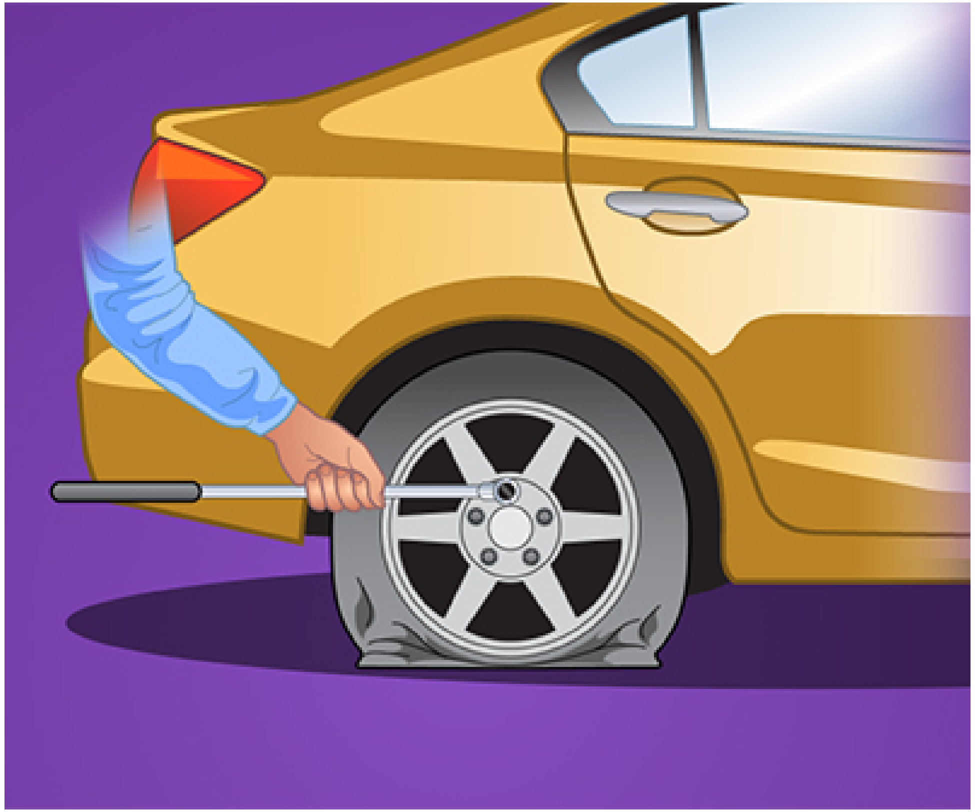
6. Calculation of Moments of Inertia

7. **Torque**

8. Relationship Between Torque and Angular Acceleration

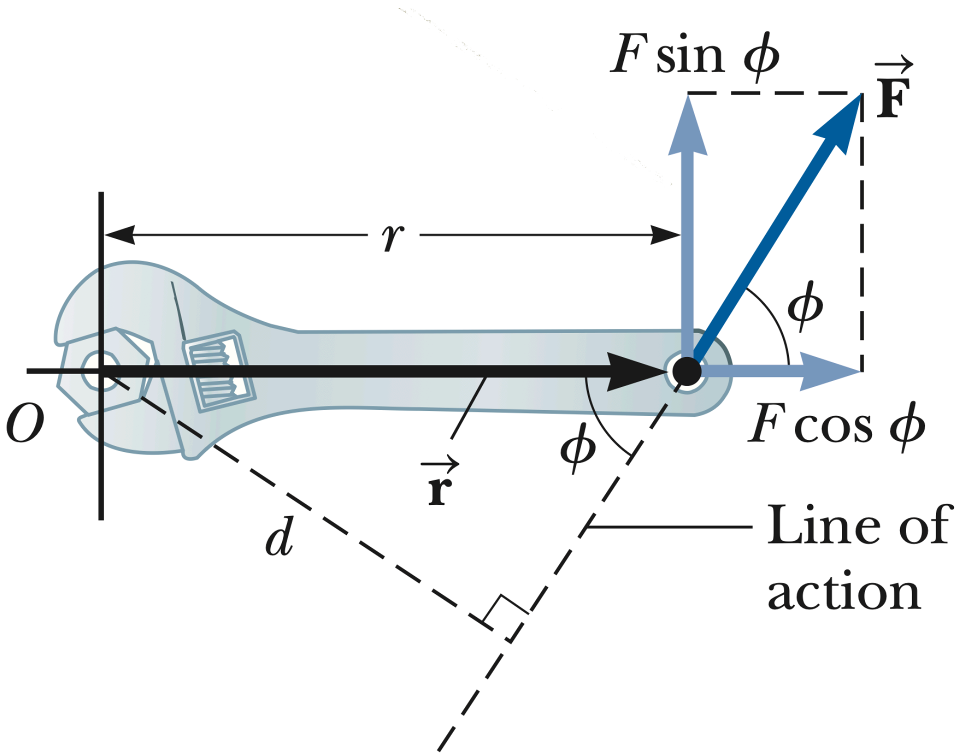
9. Work, Power, and Energy in Rotational Motion

7.1 What is Torque?



Torque τ is a measure of the strength of rotation or twisting effect around an axis.

7.1 What is Torque?

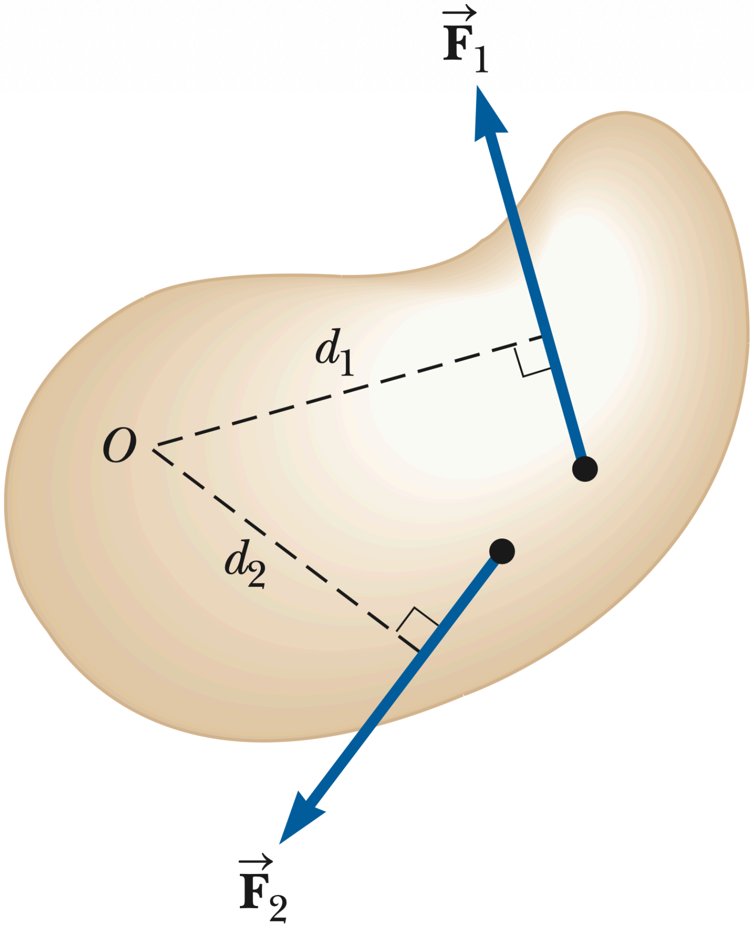


$$\tau = rF\sin\varphi = Fd$$

where the moment arm ($d = r\sin\varphi$) is the perpendicular distance from the axis of rotation to the line of action of the force.

- The SI unit of torque is the newton-meter **(N·m)**.
- Torque is a vector quantity, and its direction is given by the right-hand rule, similar to angular velocity ω.

7.2 Multiple Forces Acting on a Rigid Body

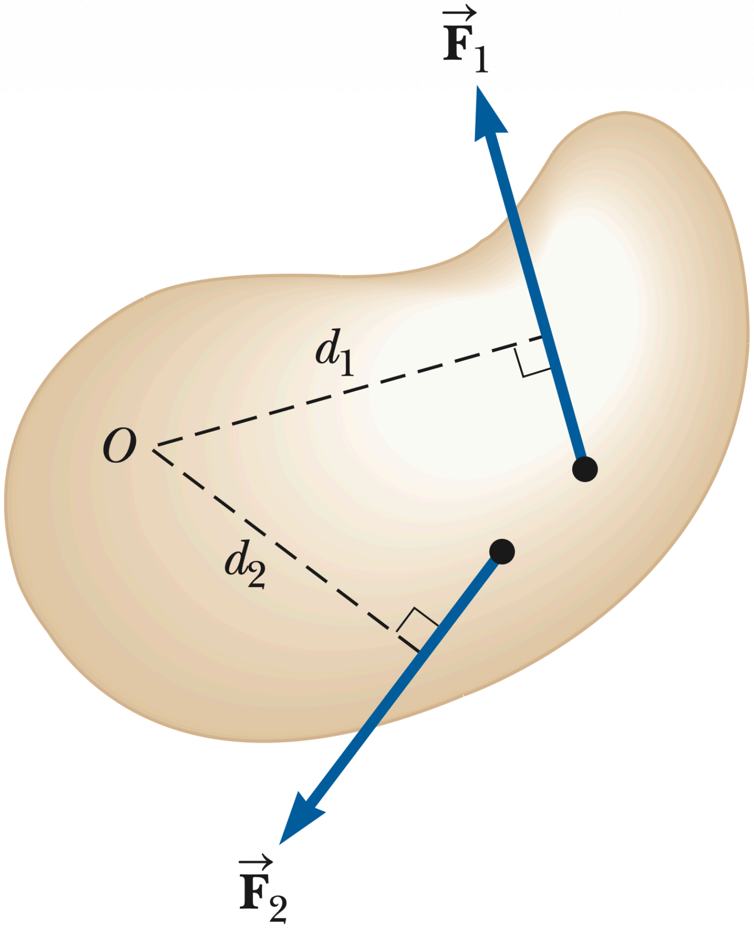


- When multiple forces act on a rigid body, the **net torque** τ_{net} about an axis is the sum of the individual torques produced by each force:

$$\tau_{net} = \sum_i \tau_i$$

- The **sign** of each torque depends on the direction of rotation it produces: counterclockwise torques are positive, and clockwise torques are negative.

7.2 Multiple Forces Acting on a Rigid Body



- Therefore, the net torque for the left figure is:

$$\tau_{\text{net}} = \tau_1 - \tau_2 = F_1 d_1 - F_2 d_2$$

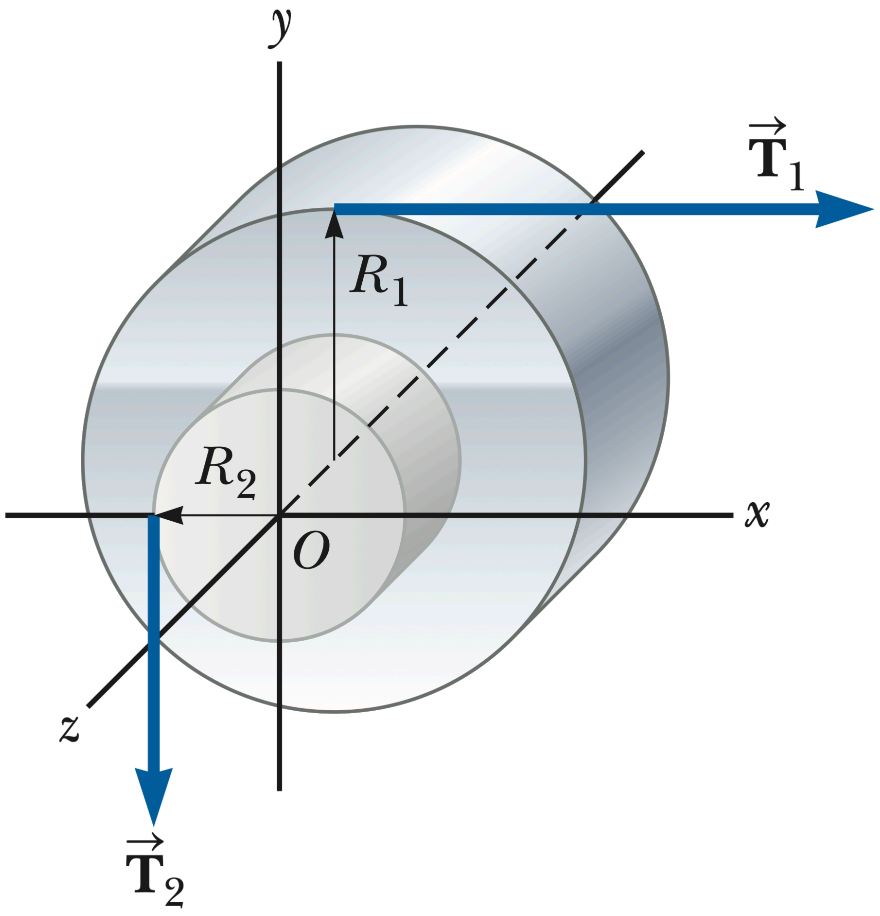
- Notice that F_2 produces a clockwise rotation, so its torque is negative.

7.2 Multiple Forces Acting on a Rigid Body

Example 7.9

A one-piece cylinder is free to rotate about the central axis along the z-axis. A rope wrapped around the drum, which has radius $R_1 = 1$ m, exerts a force $T_1 = 5$ N to the right on the cylinder. A rope wrapped around the core, which has radius $R_2 = 0.5$ m, exerts a force $T_2 = 15$ N downward on the cylinder.

What is the net torque acting on the cylinder about the rotation axis?



7.2 Multiple Forces Acting on a Rigid Body

Solution 7.9

$$t_{\text{net}} = -T_1 R_1 + T_2 R_2 = -(5\ \text{N})(1\ \text{m}) + (15\ \text{N})(0.5\ \text{m}) = 2.5\ \text{N.m}$$

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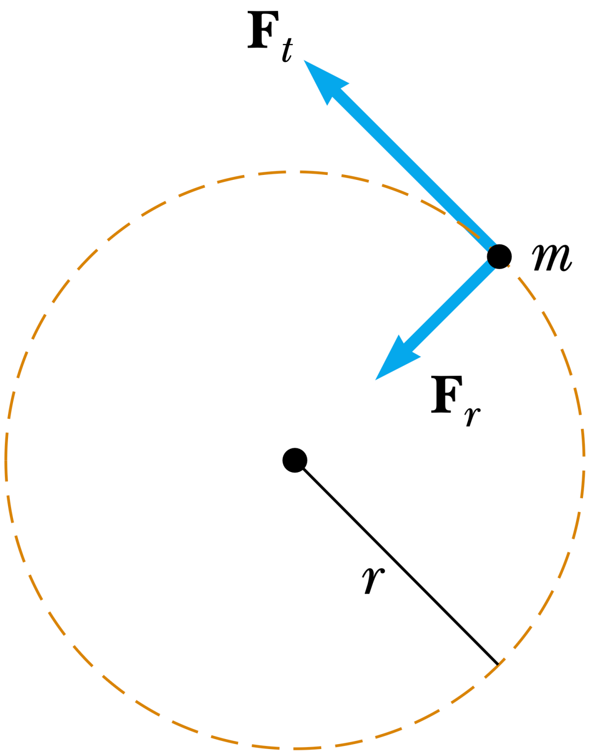
9. Work, Power, and Energy in Rotational Motion

8.1 Torque and Newton's Second Law for Rotation



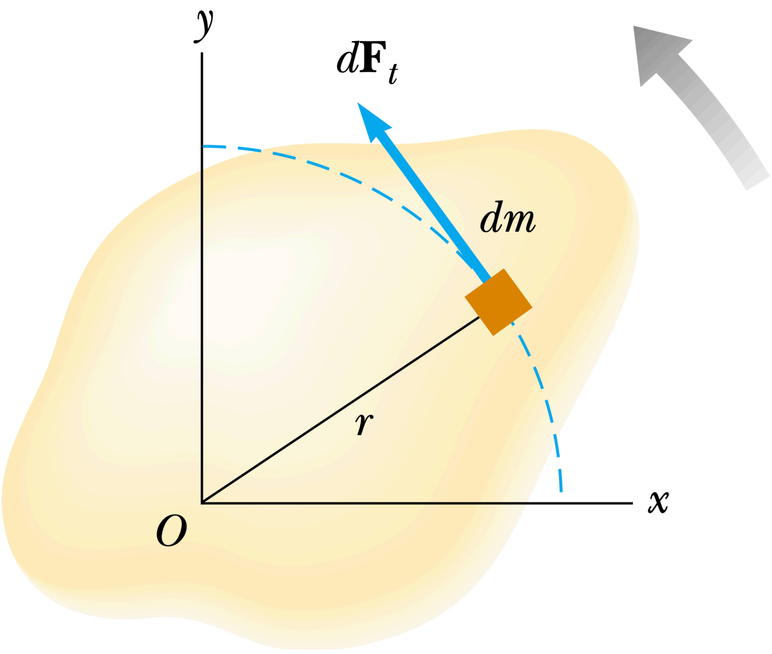
- When Forces act upon an object to rotate it about an axis, they produce a net torque τ_{net} that causes an angular acceleration α for the object.

$$\tau_{\text{net}} = F_t r = m a_t r = m(r\alpha)r = mr^2\alpha.$$

$$\boxed{\tau_{\text{net}} = I\alpha}$$

- The torque is proportional to its angular acceleration, and the proportionality constant is the moment of inertia.

8.2 Rigid Objects of Arbitrary Shape



- For a rigid object, we can divide it into tiny mass elements $\mathrm{d}m_i$ at distances r_i from the axis of rotation.

- Each mass element experiences a tangential force F_t that produces a torque τ_i about the axis:

$$\tau_i = r_i F_t = r_i(\mathrm{d}m_i)a_t$$
$$= r_i(\mathrm{d}m_i)(r_i\alpha) = r_i^2(\mathrm{d}m_i)\alpha$$

- The net torque on the object is

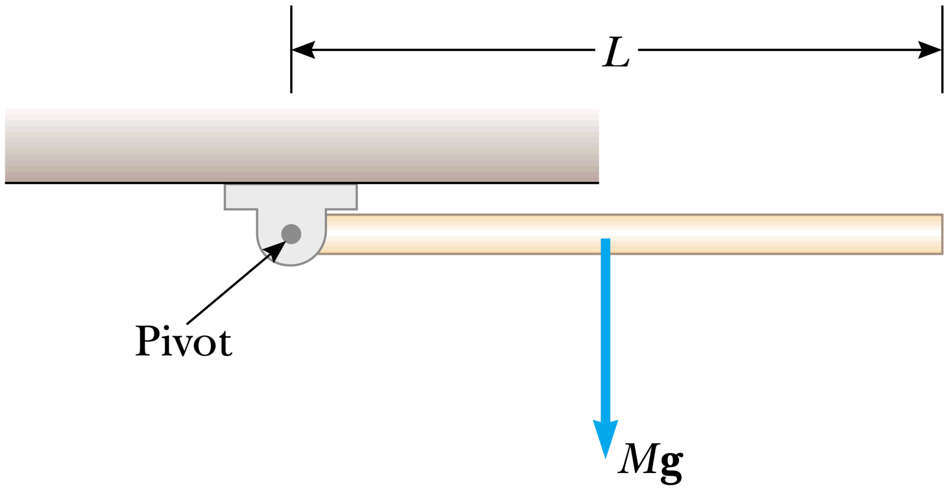
$$\tau_{\text{net}} = \sum_i r_i^2(\mathrm{d}m_i)\alpha = \alpha \sum_i r_i^2(\mathrm{d}m_i)$$
$$\tau_{\text{net}} = I\alpha$$

8.2 Rigid Objects of Arbitrary Shape

Example 8.10

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position.

What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



8.2 Rigid Objects of Arbitrary Shape

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Solution 8.10

- The magnitude of the torque about the pivot point due to the weight of the rod is:

$$\tau = Fd = (Mg)\left(\frac{L}{2}\right)$$

- Notice that we use $d = L/2$ because its the center of mass of the rod, therefore, we can assume that all its weight is concentrated at this point.

- Using $\tau = I\alpha$, we can find the angular acceleration α of the rod:

$$\alpha = \frac{\tau}{I} = \frac{Mg\frac{L}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

- The linear acceleration a of the right end of the rod is related to its angular acceleration α by the equation:

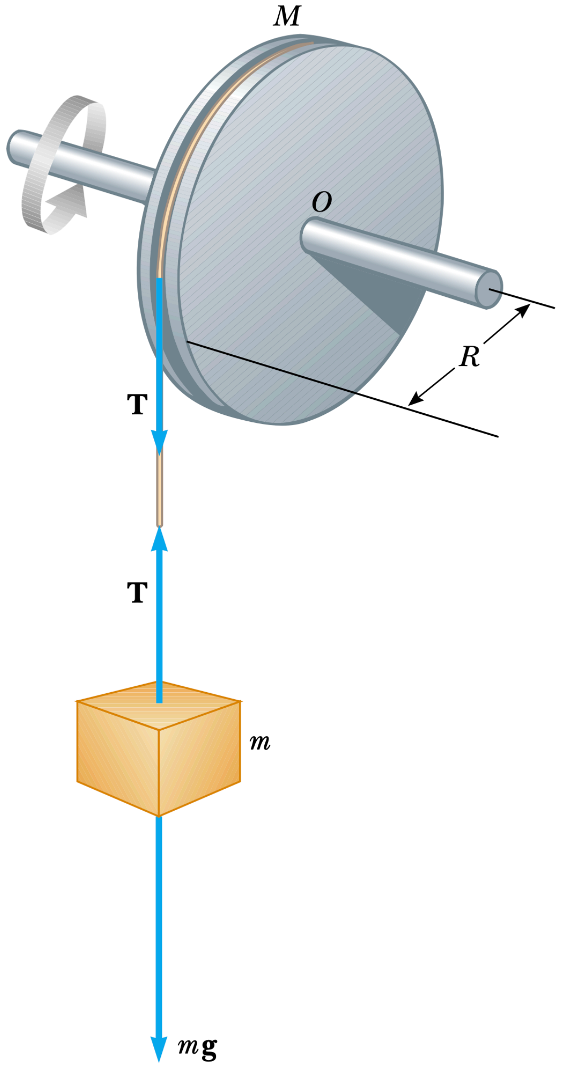
$$a = r\alpha = L\frac{3g}{2L} = \frac{3g}{2}$$

8.2 Rigid Objects of Arbitrary Shape

Example 8.11

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless horizontal axle. A light cord wrapped around the wheel supports an object of mass m.

Calculate (1) the angular acceleration of the wheel, (2) the linear acceleration of the object, and (3) the tension in the cord.



8.2 Rigid Objects of Arbitrary Shape

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Solution 8.11

- (1) For the wheel, the net torque about its axis is:

$$\tau_{\text{net}} = TR$$

- Using $\tau_{\text{net}} = I\alpha$, we have:

$$I\alpha = TR \Longrightarrow \alpha = \frac{TR}{I}$$

- (2) The linear acceleration a of the falling object is related to the angular acceleration α as:

$$a = R\alpha = \frac{TR^2}{I}$$

- Additionally, a can be found using Newton's second law:

$$\sum F = mg - T = ma$$

$$\Longrightarrow \quad a = \frac{mg - T}{m}$$

- (3) Using the previous two equations for a, we get:

$$\frac{TR^2}{I} = \frac{mg - T}{m}$$

8.2 Rigid Objects of Arbitrary Shape

Rearranging this equation to solve for the tension T in the cord gives:

$$T = \frac{mg}{1 + mR^2/I}$$

We can also rewrite α and a in a simpler form that does not require knowing T:

$$a = \frac{g}{1 + (I/mR^2)}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + (I/mR)}$$

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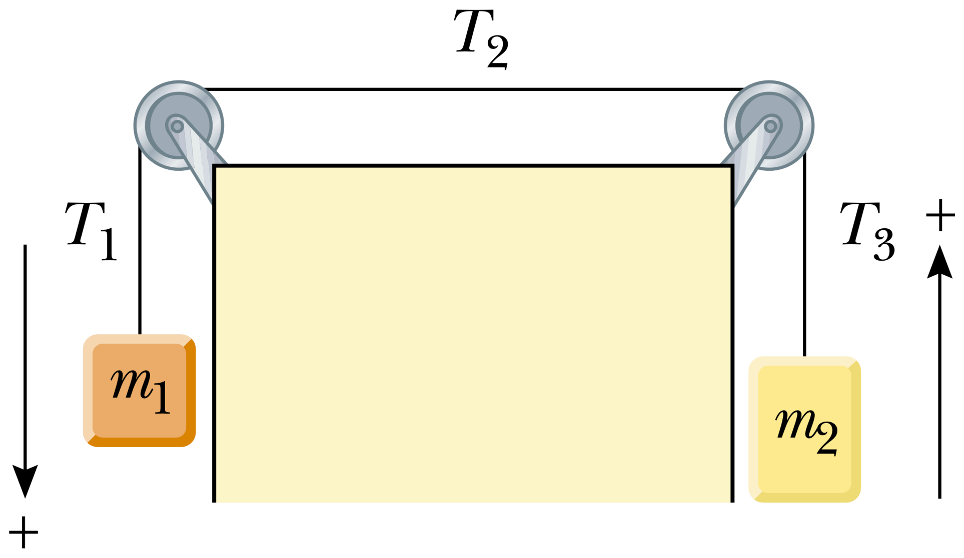
Example 8.12

Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical frictionless pulleys, each having a moment of inertia I and radius R.

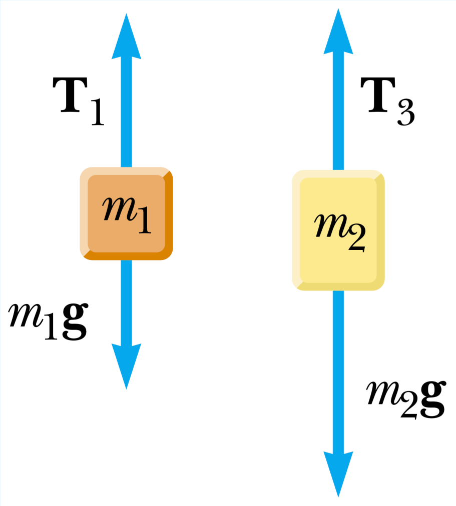
Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.



8.2 Rigid Objects of Arbitrary Shape

Solution 8.12

- First, we set the **positive** direction to be counterclockwise.
- Second, we divide the two objects

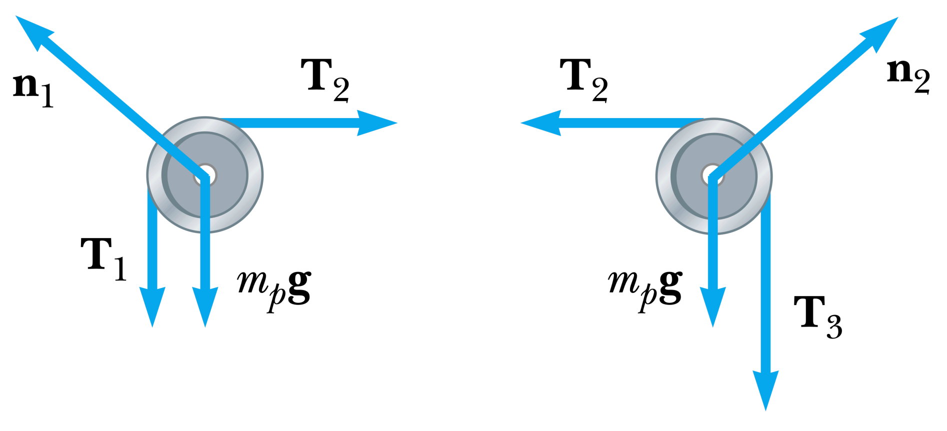


and apply Newton's second law.

$$m_1 g - T_1 = m_1 a \quad (1)$$

$$T_3 - m_2 g = m_2 a \quad (2)$$

- Next, we analyze the rotation of each pulley.



$$\tau_{\text{net}} = +T_1 R - T_2 R = I\alpha = I(a/R)$$

Therefore, the equations for the two pulleys are:

$$(T_1 - T_2)R = I(a/R) \tag{3}$$

$$(T_2 - T_3)R = I(a/R) \tag{4}$$

Solving these four equations simultaneously for a, we find:

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2I/R^2}.$$

from equation (1), we find:

$$T_1 = m_1(g - a)$$

$$T_1 = 2m_1 g\left(\frac{m_2 + I/R^2}{m_1 + m_2 + 2I/R^2}\right)$$

From equation (2), we find:

$$T_3 = m_2(g + a)$$

$$= 2m_2 g\left(\frac{m_1 + I/R^2}{m_1 + m_2 + 2I/R^2}\right)$$

Finally, from equation (3), we find:

$$T_2 = \frac{2m_1m_2 + (m_1 + m_2)(I/R^2)}{m_1 + m_2 + 2I/R^2}g.$$

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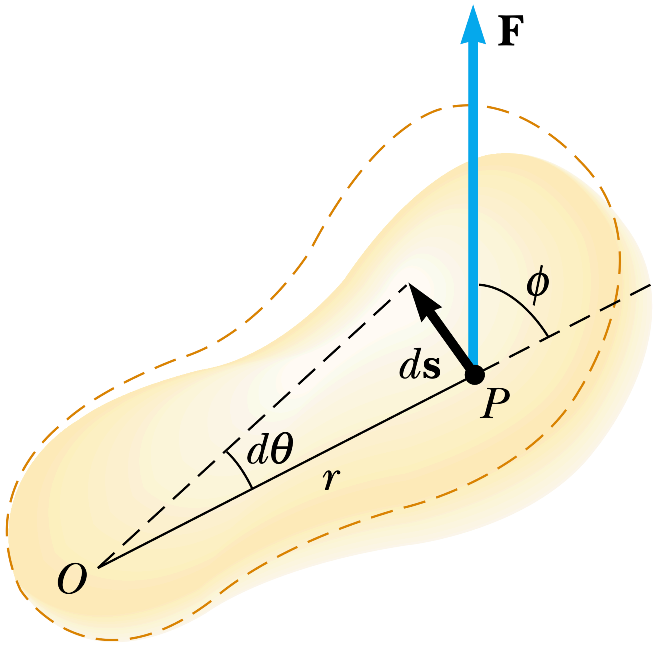
9.1 Definitions



- **Work**

$$dW = \vec{F} \cdot d\vec{s} = (F \sin\varphi)(r\, d\theta)$$

$$dW = \tau\, d\theta$$

- **Power**

$$P = \frac{dW}{dt} = \tau\omega$$

- **Work-Kinetic Energy Theorem**

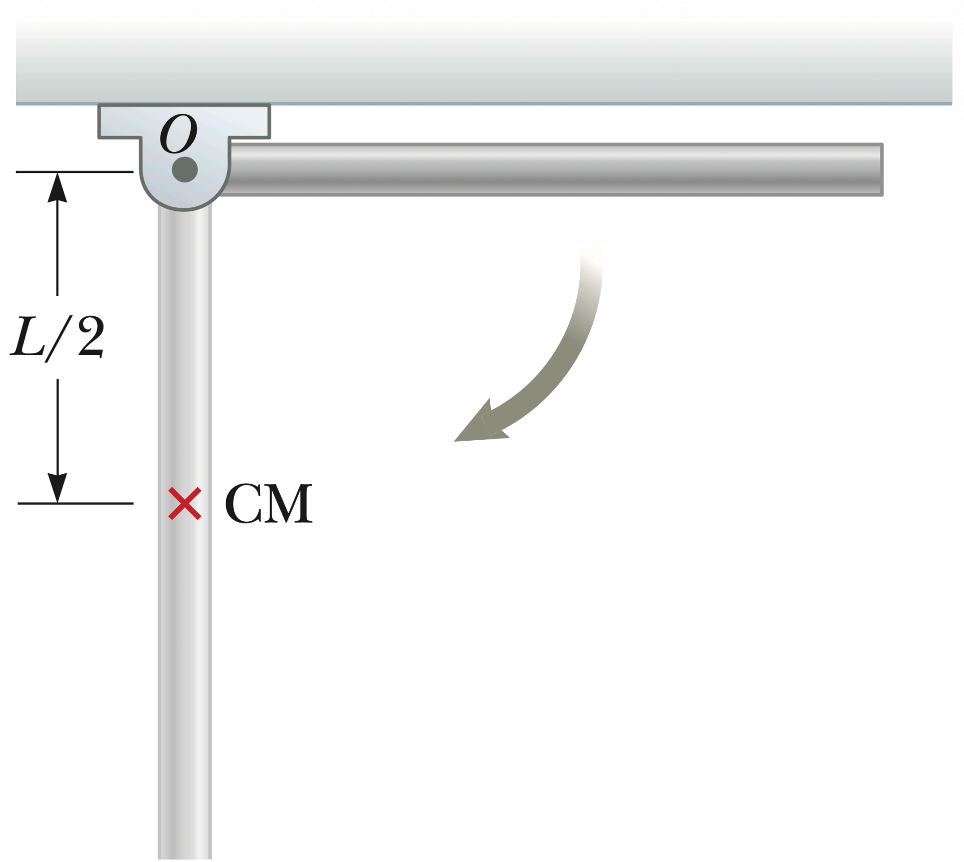
$$\sum W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 = \Delta K$$

9.1 Definitions

Example 9.13

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position.

(A) What is its angular speed when it reaches its lowest position?



9.1 Definitions

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Solution 9.13

$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$0 + Mgh = \frac{1}{2} I \omega_f^2 + 0$$

$$Mg\frac{L}{2} = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega_f^2$$

Solving for ω_f, we get:

$$\omega_f = \sqrt{\frac{3g}{L}}$$

9.1 Definitions

(B) Determine the tangential speed of the center of mass and the tangential speed of the lowest point on the rod when it is in the vertical position.

- The tangential speed of the center of mass is:

$$v_{\text{cm}} = r_{\text{cm}}\omega_f = \left(\frac{L}{2}\right)\sqrt{\frac{3g}{L}} = \sqrt{\frac{3gL}{4}}$$

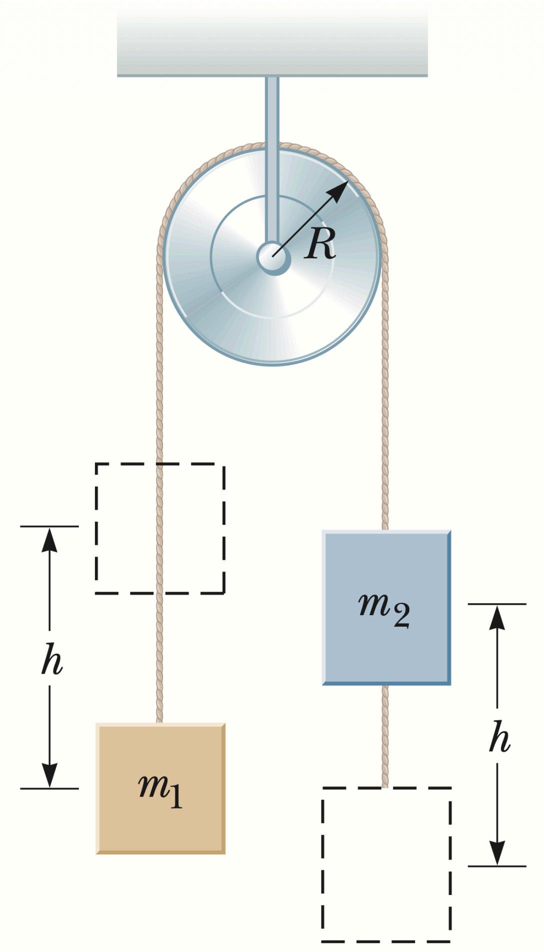
- The tangential speed of the lowest point on the rod is:

$$v_{\text{bottom}} = L\omega_f = L\sqrt{\frac{3g}{L}} = \sqrt{3gL}$$

9.1 Definitions

Example 9.14

Consider two cylinders having different masses m_1 and m_2, connected by a string passing over a pulley. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.



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Solution 9.14

$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$0 + m_2 g h = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 v_f^2 + \frac{1}{2} I \omega_f^2 + m_1 g h$$

Replacing ω_f with v_f/R, and solving for v_f, we get:

$$v_f = \sqrt{\frac{2gh(m_2 - m_1)}{m_1 + m_2 + I/R^2}}$$

and

$$\omega_f = \frac{v_f}{R} = \sqrt{\frac{2gh(m_2 - m_1)}{R^2(m_1 + m_2 + I/R^2)}}$$