Area and Volume

Mongi BLEL

King Saud University

March 25, 2024

Definition

Let Ω be an elementary subset of $\mathbb{R}^2$. The area of A is defined by:

$$\iint_{\Omega} 1dxdy.$$

Example Consider the triangle $T = \{(x, y) \in [0, 1]^2 | x + y \leq 1\}$.
We have:

$$\mathrm{Area}(T) = \int_0^1 \left(\int_0^{1-x} 1dy \right) dx = \int_0^1 (1 - x)dx = \frac{1}{2}.$$

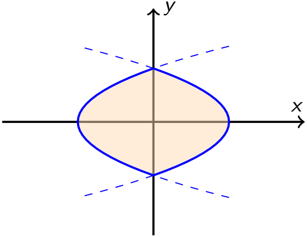
Example

Consider the disc D of center 0 and radius 1. Then

$$\begin{aligned}
\mathrm{Area}(D) &= \int_{x=-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 dy \right) dx \\
&= 2 \int_{x=-1}^{1} \sqrt{1-x^2} dx = 2 \int_{-\frac{\pi}{2}}^{\frac{x}{2}} \sqrt{1-\sin^2(\theta)} \cos(\theta) d\theta \\
&= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2\theta)) d\theta = \pi
\end{aligned}$$

Example

Consider the domain
$\Omega = \{-\sqrt{2} \le y \le \sqrt{2},\ -2 + y^2 \le x \le 2 + y^2\}$.



$$\begin{aligned}\iint_\Omega dxdy &= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-2+y^2}^{2-y^2} 1\, dx \right) dy \\ &= \frac{16}{3}\sqrt{2}.\end{aligned}$$

If $R = [-1, 1] \times [0, 2]$ and $f(x, y) = \sqrt{1 - x^2}$. The volume between R and the surface $S = \{(x, y, z) : z = f(x, y),\ (x, y) \in R\}$, is

$$V = \iint_R f(x, y) dx dy = \int_{-1}^{1} \left(\int_0^2 \sqrt{1 - x^2} dy \right) dx = \pi.$$

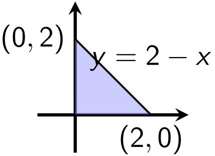
Example

The volume of the solid that lies under the graph of the function $f(x, y) = 4x^2 + y^2$ and over the region in the $xy-$plane bounded by the polygon with vertices $(0, 0), (0, 1)$ and $(2, 1)$.

$$V = \int_0^2 \int_0^1 (4x^2 + y^2) dydx = \frac{34}{3}.$$

Example

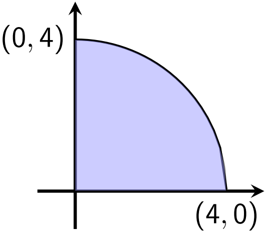
The volume of the solid in the first octant bounded by the graphs of equations
$z = 4 - x^2,\ x + y = 2,\ x = 0,\ y = 0,\ z = 0.$



$$V = \int_0^2 \int_0^{2-x} (4 - x^2) dy dx = \frac{20}{3}.$$

Example

The volume of the solid in the first octant bounded by the graphs of equations $z = x$, $x^2 + y^2 = 16$, $x = 0$, $y = 0$.



$$V = \int_0^4 \int_0^{\sqrt{16-x^2}} x dy dx = \frac{64}{3}.$$

Exercises