Department of Mathematics, College of Science

King Saud University

M-203, Midterm Examination, Semester-1, 1442H

Time: 2hours

Max. Marks-30

All questions carry equal marks

Q.1 Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}+\frac{3^n}{7^{n-1}}\right).$$

Q.2 Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{2}+\frac{1}{2n}\right)^n$$

is absolutely convergent, conditionally convergent or divergent.

Q. 3 Find the interval of convergence and radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{2^n(x+1)^n}{n+1}.$$

Q.4 Find the Maclaurin series of the function $f(x) = e^x$ and use it to approximate the integral

$$\int_0^{0.5} e^{-x^2}\,dx$$

up to four decimal places.

Q.5 Evaluate the integral

$$\int_0^1\int_{3y}^3 e^{x^2}\,dxdy$$

Q.6 Evaluate the integral

$$\iint_{\mathbf{R}} (x^2+xy)\,dA,$$

where **R** is the region bounded by the graphs of the equations

$$x=\sqrt{4-y^2},\quad y=x,\quad y=-x.$$

M - 203

Mid-term Examination (Semester I 1441/1442)

Time : 2 Hours   Max. Marks : 30

Q#1) Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}+\frac{3^n}{7^{n-1}}\right)$

[Marks: 5]

Soln: $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}+\sum_{n=1}^{\infty}\frac{3^n}{7^{n-1}}$

Let $S_1=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ we have $\frac{1}{4n^2-1}=\frac{1}{(2n-1)(2n+1)}$

$$=\frac{A}{2n-1}+\frac{B}{2n+1}$$

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$$=\frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)}$$

we get $A=-\frac{1}{2}$ and $B=\frac{1}{2}$,

$\therefore \frac{1}{4n^2-1}=\frac{1}{2}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$

$$\sum_{1}^{n}\frac{1}{4n^2-1}=\frac{1}{2}\left[\left(1-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\left(\cancel{\tfrac{1}{5}}-\tfrac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right)\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

$\therefore \sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\lim_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{2n+1}\right]=\frac{1}{2}=S_1$ 1

Also, $S_2=\sum_{n=1}^{\infty}\frac{3^n}{7^{n-1}}=\sum_{n=1}^{\infty}7\left(\frac{3}{7}\right)^n$ which is a Conv. Geom Series

$S_2=7\left[\frac{\frac{3}{7}}{\frac{4}{7}}\right]=7\left(\frac{3}{\cancel{7}}\times\frac{\cancel{7}}{4}\right)=\frac{21}{4}$ 2

$\therefore \sum_{n=1}^{\infty}\frac{1}{4n^2-1}+\sum_{n=1}^{\infty}\frac{3^n}{7^{n-1}}=S_1+S_2=\frac{1}{2}+\frac{21}{4}=\underline{\underline{\frac{23}{4}}}$ 1

Q#2) Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{2}+\frac{1}{2n}\right)^{n}$ [Marks: 5]
is absolutely convergent, conditionally convergent or divergent.

Soln. We apply absolute Root test:

$$\lim_{n\to\infty}\left|\left[(-1)^{n-1}\left(\frac{1}{2}+\frac{1}{2n}\right)^{n}\right]^{\frac{1}{n}}\right| \quad 3$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)=\frac{1}{2}<0 \Rightarrow \text{Abs. conv.}$$

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Q#3) Find the interval of convergence and radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{2^{n}(x+1)^{n}}{n+1}$.

[Marks: 5]

Soln: $$\lim_{n\to\infty}\left|\frac{2^{n+1}(x+1)^{n+1}}{(n+2)}\times\frac{n+1}{2^{n}(x+1)^{n}}\right|$$

$$=2|x+1|. \text{ For abs. conv. } |x+1|<\frac{1}{2}$$

$$\Leftrightarrow -\frac{1}{2}<x+1<\frac{1}{2}$$

$$\Leftrightarrow -\frac{3}{2}<x<-\frac{1}{2} \quad 2$$

If $x=-\frac{3}{2}$, we have $\sum_{n=0}^{\infty}\frac{2^{n}\left(-\frac{3}{2}+1\right)^{n}}{n+1}=\sum_{n=0}^{\infty}\frac{2^{n}\left(-\frac{1}{2}\right)^{n}}{n+1}$,

which is conv. by AST. 1

If $x=-\frac{1}{2}$, we have $\sum_{n=0}^{\infty}\frac{2^{n}\left(-\frac{1}{2}+1\right)^{n}}{n+1}=\sum_{n=0}^{\infty}\frac{1}{n+1}$,

which is divergent by Integral test. 1

Hence, interval of convergence: $\left[-\frac{3}{2},-\frac{1}{2}\right)$; Radius $r=\frac{-\frac{1}{2}-\left(-\frac{3}{2}\right)}{2}$
$=\frac{1}{2}$ 1

Q#4) Find the Maclaurin Series of the function $f(x)=e^x$ and use it to approximate the integral $\int_0^{0.5} e^{-x^2}dx$ upto four decimal places. [Marks: 5]

Sol. $f(x)=e^x \Rightarrow f(0)=1$; $f'(x)=e^x \Rightarrow f'(0)=1$,

$f''(x)=e^x \Rightarrow f''(0)=1$; $\cdots$; $f^{(n)}(0)=1$; $\cdots$

$\therefore f(x)=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$ (2)

$\therefore \int_0^{0.5} e^{-x^2}dx=\int_0^{0.5}\left(1-x^2+\frac{x^4}{2!}-\cdots\right)dx$ (2)

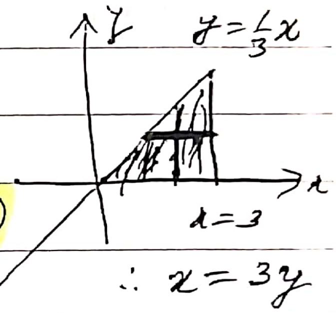
$=\left[x-\frac{x^3}{3}+\frac{x^5}{10}-\cdots\right]_0^{0.5}=0.5-\frac{0.125}{3}\approx 0.458.$ (1)

Q#5) Evaluate the integral $\int_0^1\int_{3y}^3 e^{x^2}dx\,dy$. [Marks: 5]

Sol. we reverse the order:

$$\int_0^3\int_0^{\frac{1}{3}x} e^{x^2}dy\,dx=\int_0^3 e^{x^2}\left[y\right]_0^{\frac{1}{3}x}dx \quad (4)$$

$$=\frac{1}{6}\left[e^{x^2}\right]_0^3=\frac{1}{6}(e^9-1) \quad (1)$$



$\therefore x=3y$

Put $x^2=u$; $2x\,dx=du$

$\frac{1}{6}e^u=\frac{1}{6}\left[e^{x^2}\right]_0^3$

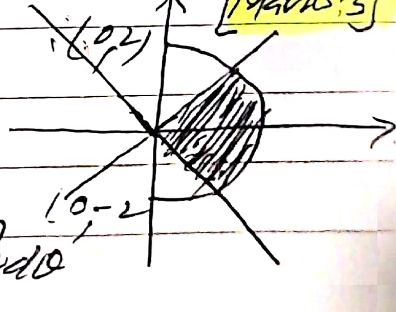
Q#6) Evaluate the integral $\iint_R (x^2+xy)\,dA$, where R is the region bounded by the graphs of the equations $x=\sqrt{4-y^2}$, $y=x$, $y=-x$ [Marks: 5]

Sol. we use polar coordinates:

$$\int_{-\pi/4}^{\pi/4}\int_0^2 r\cos\theta\,(r^2\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$=\frac{20}{3}\int_{-\pi/4}^{\pi/4}(\cos^3\theta+\cos\theta\sin\theta)\,d\theta$$

$$=\frac{20}{3}\int_{-\pi/4}^{\pi/4}\left(\cos^3\theta+\frac{1}{2}\sin 2\theta\right)d\theta$$



$$\int_{-\pi/4}^{\pi/4}\int_0^2 (r^2\cos^2\theta + r\cos\theta\, r\sin\theta)\, r\,dr\,d\theta \quad 3$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{r^4}{4}\right]_0^2 (\cos^2\theta + \cos\theta\sin\theta)\,d\theta$$

$$= 4\int_{-\pi/4}^{\pi/4}\left[\frac{1+\cos 2\theta}{2} + \frac{\sin 2\theta}{2}\right]d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta - \frac{1}{2}\cos 2\theta\right]_{-\pi/4}^{\pi/4} \quad 1$$

$$= 2\left[\frac{\pi}{4} + \frac{1}{2}\sin(\pi/2) - \cancel{\frac{1}{2}\cos\frac{\pi}{2}}\right] - 2\left[-\pi/4 + \frac{1}{2}\sin(-\pi/2) - \cancel{\frac{1}{2}\cos(-\pi/2)}\right]$$

$$= \underline{\underline{\pi + 1}} \quad 1$$