King Saud University	College of Sciences	Department of Mathematics	
First Examination	Math 132	Semester I (1447)	Time:1H30

Question 1: (12 marks)

1. Decide whether the following propositions is a tautology or a contradiction or a contingency? $[(p \leftrightarrow \neg q) \wedge (q \leftrightarrow \neg r)] \wedge (r \leftrightarrow \neg p)$. (3 marks)

2. Without using truth tables, prove that the following statement is a Tautology:
$$(r \vee \neg q) \vee (p \rightarrow q.) \quad \text{(2 marks)}$$

3. Without using truth tables, prove that the following statement is a Contradiction:
$$p \wedge (p \rightarrow q) \wedge (p \rightarrow \neg q). \quad \text{(2 marks)}$$

4. Without using truth tables, prove the following logical equivalence:
$$(p \rightarrow q) \rightarrow r \equiv (\neg r \rightarrow p) \wedge (q \rightarrow r). \quad \text{(3 marks)}$$

5. Determine the truth value of each of the following statements. (Justify your answer) (2 marks)

   (a) $\forall x \in \mathbb{R}; (x^2 - 4x + 4 \geq 0)$.

   (b) $\exists x \in \{1, 2, 3, 4\}; (2^x < x!)$.

Question 2: (13 marks)

1. Use a proof by contradiction to show that $\dfrac{\sqrt{2}-2}{3}$ is irrational. (Hint use the fact that $\sqrt{2}$ is irrational). (2 marks)

2. Let $n \in \mathbb{Z}$. Use a direct proof to show that: if n is odd then $8|(n^2 - 1)$. (2 marks)

3. Let n be an integer. Show that: n^3 is odd if and only if n is odd. (2 marks)

4. Use mathematical induction to show that $5n^2 - 3n$ is even for every nonnegative integer n. (3 marks)

5. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_0 = 2 \\ u_1 = 6 \\ u_{n+1} = 2u_n + 3u_{n-1}; \quad n \geq 1 \end{cases}$

   Use mathematical induction to prove the following statement:
$$u_n = 2 \times 3^n, \quad \text{for each integer } n, \text{ with } n \geq 0. \quad \text{(4 marks)}$$

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1) 3

p	q	r	$\neg q$	$p \leftrightarrow \neg q$	$\neg r$	$q \leftrightarrow \neg r$	A	$\neg p$	$r \leftrightarrow \neg p$	B
T	T	T	F	F	F	F	F	F	F	F
T	T	F	F	F	T	T	F	F	T	F
T	F	T	T	T	F	T	T	F	F	F
T	F	F	T	T	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	T	F
F	T	F	F	T	T	T	T	T	F	F
F	F	T	T	F	F	T	F	T	T	F
F	F	F	T	F	T	F	F	T	F	F

a contradiction.

2) 2

$$(r \vee \neg q) \vee (p \to q) \equiv (r \vee \neg q) \vee (\neg p \vee q)$$
$$\equiv (r \vee \neg p) \vee (\neg q \vee q)$$
$$\equiv (r \vee \neg p) \vee T \equiv T.$$

3) 2

$$p \wedge (p \to q) \wedge (p \to \neg q) \equiv p \wedge (p \to (q \wedge \neg q))$$
$$\equiv p \wedge (p \to F)$$
$$\equiv p \wedge (\neg p \vee F).$$
$$\equiv p \wedge \neg p \equiv F.$$

4) 3

$$(p \to q) \to r \equiv \neg(\neg p \vee q) \vee r$$
$$\equiv (p \wedge \neg q) \vee r \equiv (p \vee r) \vee (\neg q \vee r)$$
$$\equiv (\neg(\neg r) \vee p) \vee (\neg q \vee r)$$
$$\equiv (\neg r \to p) \vee (q \to r).$$

5) True.

a) $x^2 - 4x + 4 = (x-2)^2 \geq 0$. 1

b) True $\underline{n=4}$: $2^4 = 16 < 4! = 24$.

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1) By contradiction, we suppose that $\frac{\sqrt{2}-2}{3} \in \mathbb{Q}$.

$\Rightarrow \frac{\sqrt{2}-2}{3} = \frac{a}{b}$; $a, b \in \mathbb{Z}$; $b \neq 0$.

2 $\Rightarrow \sqrt{2} - 2 = \frac{3a}{b}$.

$\Rightarrow \sqrt{2} = \frac{\overbrace{3a+2b)}^{\in \mathbb{Z}}}{\underbrace{b}_{\in \mathbb{Z},\ \neq 0}} \in \mathbb{Q}$ contradiction

$\Rightarrow \frac{\sqrt{2}-2}{3} \notin \mathbb{Q}$.

2) we assume that n is odd $\Rightarrow n = 2k+1$, $k \in \mathbb{Z}$.

$\Rightarrow n^2 = 4k^2 + 4k + 1$

$\Rightarrow n^2 - 1 = 4k^2 + 4k$

$= 4(k^2 + k)$

$= 4(\underbrace{k(k+1)}_{\text{even}})$; $k(k+1)$ is even $\Rightarrow k(k+1) = 2t$, $t \in \mathbb{Z}$

$= 8t \Rightarrow 8 \mid (n^2 - 1)$.

3) • if n^3 is odd, then n is odd.

By contraposition, we show that: if n is even, then n^3 is even.

1 we assume that n is even, $\Rightarrow n = 2k$; $k \in \mathbb{Z}$.

$\Rightarrow n^3 = (2k)^3 = 2(\underbrace{4k^3}_{\in \mathbb{Z}})$.

$\Rightarrow n^3$ is even.

• if n is odd, then n^3 is odd.

1 we assume that n is odd $\Rightarrow n = 2k+1$, $k \in \mathbb{Z}$.

$\Rightarrow n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$

$= 2(\underbrace{4k^3 + 6k^2 + 3k}_{\in \mathbb{Z}}) + 1$

$\Rightarrow n^3$ is odd.

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4) $P(n): (5n^2 - 3n)$ is even; $\forall n \geq 0$.

B.S: $P(0): 5\times 0 - 3\times 0 = 0$ is even.

I.S: let $k \geq 0$, we prove that $P(k) \to P(k+1)$.

we assume that $P(k)$ is true $5k^2 - 3k$ is even.

$\Rightarrow 5k^2 - 3k = 2a;\ a \in \mathbb{Z}$.

we prove that $P(k+1)$ is true

$5(k+1)^2 - 3(k+1)$ is even.

$$\begin{aligned} 5(k+1)^2 - 3(k+1) &= 5(k^2 + 2k + 1) - 3(k+1) \\ &= 5k^2 + 10k + 5 - 3k - 3 \\ &= 5k^2 - 3k + 2 + 10k \\ &= 2a + 10k + 2 \\ &= 2(\underbrace{a + 5k + 1}_{\in \mathbb{Z}}) \end{aligned}$$

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$\Rightarrow 5(k+1)^2 - 3(k+1)$ is even

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 0;\ P(n)$ is true.

5) $P(n): U_n = 2\times 3^n;\quad \forall n \geq 0$.

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B.S: $P(0): u_0 = 2\times 3^0 = 2$. True.

$P(1): u_1 = 2\times 3^1 = 6$ True.

I.S: let $k \geq 1$; we assume that $P(0), P(1), \ldots, P(k)$ are true and we prove that $P(k+1)$ is true.

$P(k+1):\ u_{k+1} = 2\times 3^{k+1}$.

$u_{k+1} = 2u_k + 3u_{k-1}$

$P(k)$ true $\Rightarrow u_k = 2\times 3^k$

$P(k-1)$ true $\Rightarrow u_{k-1} = 2\times 3^{k-1}$.

$$\begin{aligned} u_{k+1} &= 2\times 2\times 3^k + 3\times 2\times 3^{k-1} \\ &= 4\times 3^k + 2\times 3^k = 2\times 3^k(2+1) \\ &= 2\times 3^{k+1} \end{aligned}$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 0;\ P(n)$ is true