King Saud University	College of Sciences		Department of Mathematics
First Examination	Math 132	Semester 2 (1446)	Time:1H30

Question 1: (10 marks)

1. Decide whether the following propositions is a tautology or a contradiction or a contingency?

$$[p \to q) \wedge (q \to r)] \vee \neg r. \quad (3 \text{ marks})$$

2. Without using truth tables, prove that the following conditional statement is a Contradiction:

$$\neg [p \to (q \vee r)] \wedge \neg p. \quad (3 \text{ marks})$$

3. Without using truth tables, prove the following logical equivalence:

$$\neg p \to (p \wedge \neg q) \equiv p. \quad (2 \text{ marks})$$

4. Determine the truth value of each of the following statements. (Justify your answer) (2 marks)

   (a) $\forall x \in \mathbb{Z}; (x^2 > 0)$.

   (b) $\exists x \in \mathbb{R}; (x^2 = -3)$.

Question 2: (8 marks)

1. Let a and b be two integers such that $a \geq 2$. Use a proof by contradiction to show that $a \not| b$ or $a \not| (b+1)$. (3 marks)
2. Let a be an integer. Use a proof by contraposition to show that:
   If $a^2 - 2a + 7$ is even, then a is odd. (3 marks)
3. Let x, y and z be three integers. Give a direct proof to show that :
   If $x|y$ and $y|z$, then $x|z$. (2 marks)

Question 3: (7 marks)

1. Use mathematical induction to prove the following statement:

$$5 + 10 + 15 + \cdots + 5n = \frac{5n(n+1)}{2}, \quad \text{for each integer } n, \text{ with } n \geq 1. \quad (3 \text{ marks})$$

2. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_0 = 0 \\ u_1 = 1 \\ u_{n+1} = 5u_n - 6u_{n-1}; \quad n \geq 1 \end{cases}$

   Use mathematical induction to prove the following statement:

$$u_n = 3^n - 2^n, \quad \text{for each integer } n, \text{ with } n \geq 0. \quad (4 \text{ marks})$$

Mid1 (Math 132)
462.

Q1) 10

1) $A = (p \to q) \wedge (q \to r)$; $B = A \vee \neg r$.

p	q	r	$p \to q$	$q \to r$	A	$\neg r$	B
T	T	T	T	T	T	F	T
T	T	F	T	F	F	T	T
T	F	T	F	T	F	F	F
T	F	F	F	T	F	T	T
F	T	T	T	T	T	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	T	T

3

Contingency.

2)
$$\neg(p \to (q \vee r)] \wedge \neg p \equiv \neg[\neg p \vee (q \vee r)] \wedge \neg p.$$
$$\equiv p \wedge \neg(q \vee r) \wedge \neg p.$$
$$\equiv (p \wedge \neg p) \wedge \neg(q \vee r)$$
$$\equiv F \wedge \neg(q \vee r) \equiv F.$$

3

3)
$$\neg p \to (p \wedge \neg q) \equiv p \vee (p \wedge \neg q)$$
$$\equiv p.$$

2

4) a) false $0^2 \not> 0$. 1

b) $\exists x \in \mathbb{R};\ x^2 = -3$; false 1

because $\forall x \in \mathbb{R};\ x^2 \geq 0$.

$Q_2)$ (8)

1) let $a, b \in \mathbb{Z}$, such that $a \geq 2$.

By contradiction we assume that $a | b$ and $a | (b+1)$

$\Rightarrow b = q_1 a$ and $b+1 = q_2 a$; $q_1, q_2 \in \mathbb{Z}$.

(3) $\Rightarrow q_2 a - q_1 a = 1 \Rightarrow a(q_2 - q_1) = 1$.

$\Rightarrow q_2 - q_1 = \frac{1}{a} \notin \mathbb{Z}$ because $a \geq 2$

a contradiction.

2) by contraposition, we prove that:

if a is even, then $a^2 - 2a + 7$ is odd

(3) assume that a is even $\Rightarrow a = 2k$; $k \in \mathbb{Z}$.

$\Rightarrow a^2 - 2a + 7 = 4k^2 - 4k + 7$.

$= 2(\underbrace{2k^2 - 2k + 3}_{\in \mathbb{Z}}) + 1 \Rightarrow a^2 - 2a + 7$ is odd.

3) let $x, y, z \in \mathbb{Z}$; assume that $x | y$ and $y | z$

$\Rightarrow y = q_1 x$ and $z = q_2 y$ $q_1, q_2 \in \mathbb{Z}$.

(2) $\Rightarrow z = q_2 y = \underbrace{q_2 q_1}_{\in \mathbb{Z}} x$

$\Rightarrow x | z$.

Q_3 7

1) $\cdot P(n): 5+10+15+\dots+5n = \frac{5n(n+1)}{2}$. $\forall n \geq 1$.

B.S. $P(1)$: L.H.S $= 5$

R.H.S $= \frac{5\times1(1+1)}{2} = 5$

I.S: let $k \geq 1$, we prove that $P(k) \to P(k+1)$

Assume that $P(k)$ true

$5+10+\dots+5k = \frac{5k(k+1)}{2}$.

we prove that $P(k+1)$ is true

3

$5+10+\dots+5(k+1) = \frac{5(k+1)(k+2)}{2}$

$5+10+\dots+5k+5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$

$= \frac{5k(k+1)+5(k+1)2}{2}$

$= \frac{5(k+1)(k+2)}{2} \Rightarrow P(k+1)$ is true.

2) $P(n): u_n = 3^n - 2^n$, $\forall n \geq 1$

B.S. $P(0): u_0 = 1 - 1 = 0$ true.

$P(1): u_1 = 3 - 2 = 1$ true.

I.S. let $k \geq 1$; we assume that $P(0), P(1), \dots, P(k)$ are true

and we prove that $P(k+1)$ is true.

$u_{k+1} = 3^{k+1} - 2^{k+1}$??

$u_{k+1} = 5u_k - 6u_{k-1}$, $\quad P(k)$ true $\Rightarrow u_k = 3^k - 2^k$.

$P(k-1)$ true $\Rightarrow u_{k-1} = 3^{k-1} - 2^{k-1}$.

$u_{k+1} = 5\times3^k - 5\times2^k - 6\times3^{k-1} + 6\times2^{k-1}$.

$= 5\times3^k - 2\times3^k - 5\times2^k + 3\times2^k$

4 $= 3^k(5-2) - 2^k(5-3) = 3\times3^k - 2\times2^k = 3^{k+1} - 2^{k+1}$

$\Rightarrow P(k+1)$ true.