King Saud University	College of Sciences		Department of Mathematics
Final Examination	Math 132	Semester 2 (1446)	Time: 3 Hours

Question 1: (9 marks)

1. Determine whether the following statement is a tautology, a contradiction, or a contingency.
$$(p \leftrightarrow q) \vee (q \leftrightarrow r) \vee (r \leftrightarrow p). \quad \text{(2 marks)}$$
2. Let m and n be integers. Use a direct proof to show that if both $m-1$ and $n+2$ are divisible by 3, then $mn + 2m - n + 7$ is divisible by 9. (3 marks)
3. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_1 = 0 \\ u_2 = 8 \\ u_{n+1} = \dfrac{u_n u_{n-1} - 48}{4}; \quad n \geq 1 \end{cases}$

   Use mathematical induction to show that $4|u_n$ for each integer n with $n \geq 1$. (4 marks)

Question 2: (15 marks)

1. Let R be the relation on the set $\mathbb{Z}$ defined by
$$a, b \in \mathbb{Z};\ aRb \iff a > b - 2$$
Decide whether the relation R is reflexive, symmetric, antisymmetric or transitive. Justify your answers. (4 marks)
2. Let E be the relation on the set $\mathbb{R}$ defined by
$$x, y \in \mathbb{R};\ xEy \iff (x^2 - y^2) \in \mathbb{Z}$$
   (a) Show that E is an equivalence relation. (3 marks)
   (b) Show that $[x] = [-x]$. (1 mark)
   (c) Is 5 in $[\sqrt{5}]$. (1 mark)
3. Let $P = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d), (e,e)\}$ be a relation on the set $\{a, b, c, d, e\}$.
   (a) Represent P with a digraph. (1 mark)
   (b) Prove that P is a partial order. (3 marks)
   (c) Is P a total order. Justify your answer. (1 mark)
   (d) Represent P with a Hasse diagram. (1 mark)

Question 3: (13 marks)

1. Let X and Y be two sets such that: $X - Y = \{0, 1, 2\}$, $Y - X = \{-2, -1\}$, and $X \cup Y = \{-3, -2, -1, 0, 1, 2, 3\}$. Find X and Y. (1 mark)
2. Let $A = \{x, y, z, t\}$ and $B = \{x, y\}$ be two sets.
   (a) Find $(A - B) \times B$. (2 marks)
   (b) Find $A \times \{\emptyset\}$. (1 mark)
3. Let f be the function from $C := \{x, y, z\}$ to $D := \{a, b, c\}$ defined by $f(x) = c$, $f(y) = a$, and $f(z) = b$.
   (a) Find $f(\{x, y\})$ and $f(\{y, z\})$. (1 mark)
   (b) Find $f^{-1}(\{a\})$ and $f^{-1}(\{a, b\})$. (1 mark)
   (c) Decide whether f is one to one or onto. Justify your answers. (2 marks)
4. Let g be a function from $\mathbb{Z}$ to $\mathbb{R}$ and let h be a function from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(x) = x - 1$ and $h(x) = 2x + 2$.
   (a) Find $g \circ h$ and $h \circ g$ if it is possible. (2 marks)
   (b) Prove that h is a one to one correspondence function. (2 marks)
   (c) Find $h^{-1}(x)$, for all $x \in \mathbb{R}$. (1 mark)

Question 4: (3 marks)

1. Give the cardinal of each of the following sets.
   (a) $A_1 := \mathbb{Z} \cap [0, 1]$. (1 mark)
   (b) $A_2 := [0, \infty) \cup \mathbb{Q}^+$. (1 mark)
2. Give three sets A, B and C such that $|A| < |B| < |C|$. (1 mark)

Answer final

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Q1) 9

1)

p	q	r	$p\leftrightarrow q$	$q\leftrightarrow r$	$r\leftrightarrow p$	A
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	F	T	F	T
F	T	F	F	F	T	T
F	F	T	T	F	F	T
F	F	F	T	T	T	T

tautology.

2

2) let $m, n \in \mathbb{Z}$; $3|(m-1)$ and $3|(n+2)$.

$\Rightarrow m-1 = 3q_1$; $n+2 = 3q_2$, $q_1, q_2 \in \mathbb{Z}$.

$\Rightarrow m = 3q_1+1$ and $n = 3q_2-2$.

3

$\Rightarrow mn + 2m - n + 7 = (3q_1+1)(3q_2-2) + 6q_1 + 2 - 3q_2 + 2 + 7$.

$= 9q_1q_2 - 6q_1 + 3q_2 - 2 + 6q_1 + 2 - 3q_2 + 9$.

$= 9q_1q_2 + 9 = 9(q_1q_2+1)$, $\Rightarrow 9|(mn+2m-n$

3) $P(n)$: $4|u_n$, $\forall n \geq 1$.

B.S: $P(1)$: $4|0 \Rightarrow P(1)$ true.

$P(2)$: $4|8 \Rightarrow P(2)$ true.

I.S. let $k \geq 2$; we assume that $P(1), P(2), \dots, P(k)$ are true and we prove that $P(k+1)$ is true $\Rightarrow$ $P(k+1)$: $4|u_{k+1}$.

4

$u_{k+1} = \dfrac{u_k \cdot u_{k-1} - 48}{4}$, $P(k)$ true $\Rightarrow 4|u_k \Rightarrow u_k = 4a$.

$P(k-1)$ true $\Rightarrow 4|u_{k-1} \Rightarrow u_{k-1} = 4b$ $a, b \in \mathbb{Z}$.

$= \dfrac{4a4b-48}{4}$

$= 4(ab) - 12 = 4\underbrace{(ab-3)}_{\in\mathbb{Z}} \Rightarrow 4|u_{k+1} \Rightarrow P(k+1)$ is true

$\Rightarrow P(n)$ is true $\forall n \geq 1$.

2) 15

1)

- $a \in \mathbb{Z}$; $a - a = 0 > -2 \Rightarrow a > a-2$.
  $\Rightarrow aRa \Rightarrow R$ is reflexive

- $2 > 3-2$ and $3 > 2-2$
  $\Rightarrow 2R3$ and $3R2 \Rightarrow$ R is not antisymmetric.

- $a=3$; $b=-3$; $3 > -3-2 \Rightarrow 3R-3$, $-3 \not> 3-2 \Rightarrow -3 \not R 3$ $\Big\} \Rightarrow$ R is not symmetric

- $-3R0$ $(-3 > 0-2)$.
  $0R1$ $(0 > 1-2)$.
  but $-3 \not R 1$ $(-3 \not> 1-2)$.

4

2) a) $x \in \mathbb{R}$; $x^2 - x^2 = 0 \in \mathbb{Z} \Rightarrow xEx$
$\Rightarrow E$ is reflexive.

$x, y \in \mathbb{R}$; $xEy \Rightarrow (x^2 - y^2) \in \mathbb{Z} \Rightarrow (y^2 - x^2) \in \mathbb{Z}$.
$\Rightarrow yEx \Rightarrow E$ is symmetric

3

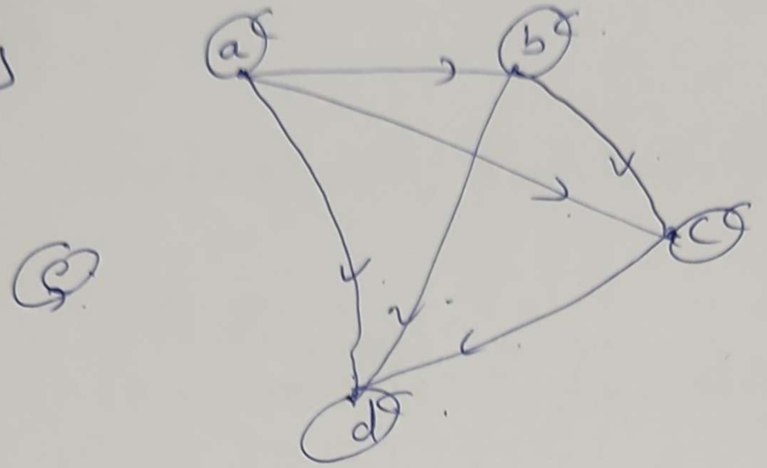
$x, y, z \in \mathbb{R}$; xEy and $yEz \Rightarrow (x^2 - y^2) \in \mathbb{Z}$ and $(y^2 - z^2) \in \mathbb{Z}$
$\Rightarrow (x^2 - z^2) \in \mathbb{Z}$
$\Rightarrow xEz \Rightarrow E$ is transitive

$\Rightarrow E$ is an equivalence relation.

b). $x^2 - (-x)^2 = x^2 - x^2 = 0 \in \mathbb{Z} \Rightarrow xE(-x)$
$\Rightarrow [x] = [-x]$. 1

c) $5^2 - (\sqrt{5})^2 = 25 - 5 = 20 \in \mathbb{Z}$.
$\Rightarrow 5E\sqrt{5} \Rightarrow 5 \in [\sqrt{5}]$ 1

3) a)



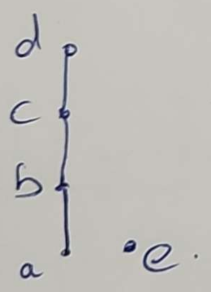
1

b) • $(a,a), (b,b), (c,c), (d,d), (e,e) \in P \Rightarrow P$ is reflexive.

• $(a,b) \in P, (b,a) \notin P$; $(a,c) \in P, (c,a) \notin P$; $(a,d) \in P, (d,a) \notin P$; $(b,c) \in P, (c,b) \notin P$. $(b,d) \in P, (d,b) \notin P, (c,d) \in P, (d,c) \notin P \Rightarrow P$ is antisymmetric.

• $\left.\begin{array}{l} (a,b) \in P; (b,c) \in P \Rightarrow (a,c) \in P \\ (a,b) \in P, (b,d) \in P \Rightarrow (a,d) \in P \\ (b,c) \in P, (c,d) \in P \Rightarrow (b,d) \in P \end{array}\right\} \Rightarrow P$ is transitive. 3

c) No; a, e are incomparable. 1



1

Q_3) 13

1) $X = \{0, 1, 2, -3, 3\}$; $Y = \{-2, -1, -3, 3\}$. 1

2) a) $A - B = \{z, t\}$.
$(A-B) \times B = \{(z,x), (z,y), (t,x), (t,y)\}$. 2

b). $A \times \{\phi\} = \{(x,\phi), (y,\phi), (z,\phi), (t,\phi)\}$. 1

3) a) $f(\{x,y\}) = \{a,c\}$; $f(\{y,z\}) = \{a,b\}$. 1

b) $f^{-1}(\{a\}) = \{y\}$; $f^{-1}(\{a,b\}) = \{y,z\}$ 1

c). $f(x) \neq f(y) = f(z) \Rightarrow f$ is one to one.

2 $f^{-1}(\{a\}) = \{y\}$; $f^{-1}(\{b\}) = \{z\}$; $f^{-1}(\{c\}) = \{x\}$.
$\Rightarrow f$ is onto.

4) a) $(g \circ h)(x) = g(h(x)) = 2x + 2 - 1 = 2x + 1$ 2
$(h \circ g)(x) = h(g(x)) = 2(x-1) + 2 = 2x$.

b) $x, y \in \mathbb{R}$;
$2x + 2 = 2y + 2 \Rightarrow 2x = 2y \Rightarrow x = y \Rightarrow h$ is one to one.
let $y \in \mathbb{R}$; $\exists ? x \in \mathbb{R}$ st; $h(x) = y \Rightarrow 2x + 2 = y$
$\Rightarrow x = \frac{y-2}{2}$. 2
$\Rightarrow \exists x = \frac{y-2}{2} \in \mathbb{R}$ st $h(x) = y \Rightarrow h$ is onto.

c) $h^{-1}(y) = \frac{y-2}{2}$. 1

Q4) 3

1)

a) $A_1 = \mathbb{Z} \cap [0,1]$

$= \{0,1\} =$

$|A_1| = 2$ 1

b) $A_2 = [0,\infty) \cap \mathbb{Q}^+ \subseteq \mathbb{Q}^+$.

$\Rightarrow |A_2| = \aleph_0$. 1

2)

$A = \mathbb{Z}^+$,

$B = \mathcal{P}(\mathbb{Z}^+)$

$C = \mathcal{P}(B)$. 1

5)