King Saud University	College of Sciences	Department of Mathematics
Final Examination	Math 132	Semester I(1442)	Time: 3H

Calculators are not allowed

The Examination contain 2 pages

Exercise 1:(15 pts)

1. Prove that the following statement is a tautology: (3 pts)

$$(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$$

2. Without using truth tables, prove the following logical equivalence: (3 pts)

$$\neg(p \rightarrow q) \rightarrow q \equiv p \rightarrow q$$

3. Let m, n and p are integers.
   Prove that: if $m+n$ and $n+p$ are even then $m+p$ is even. (2 pts)

4. Use mathematical induction to prove that $2+4+6+\cdots+2n = n(n+1)$, whenever n is a positive integer. (3 pts)

5. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined as follows:
   $a_0 = 2, a_1 = 6$, and $a_{n+1} = a_n + a_{n-1}; \forall n \geq 1$.
   Use mathematical induction to prove that $2|a_n$, for all integers $n \geq 0$. (4 pts)

Exercise 2:(11 pts)

1. Let R be the relation defined on $\mathbb{Z}$ by

$$m, n \in \mathbb{Z}, mRn \Longleftrightarrow m+n \text{ is even .}$$

   Decide whether the relation R is reflexive, symmetric, anti-symmetric, or transitive. (4 pts)

2. Let E be the relation defined on the set $S = \{1,2,3,4,5,6\}$ by:
   $E = \{(1,1),(1,3),(1,5),(2,2),(2,6),(3,1),(3,3),(3,5),(4,4),(5,1),(5,3),(5,5),(6,2),(6,6)\}$.

   (a) Represent E with a digraph. (2 pts)

   (b) Prove that E is an equivalence relation. (3 pts)

   (c) Find all distinct equivalence classes of E. (2 pts)

Exercise 3:(10 pts)

1. Consider the following three sets $X := \{1, 2, 3, 4\}, Y := \{2, 3\}$, and $Z := \{(1, 2), (1, 4), (2, 2), (2, 4), (4, 4), (2, 3)\}$. Find the following sets:
   (a) $X \times (Y \setminus X)$. (1 pts)
   (b) $(Y \times X) \cap Z$. (1 pts)
   (c) $(Y \times X) \setminus Z$. (1 pts)
2. Consider the sets $A := \{a, b, c, d\}$ and $B := \{0, 2, 4, 6\}$, and the function $f : A \longrightarrow B$ defined by: $f(a) = f(d) = 4$, $f(b) = 6$ and $f(c) = 0$.
   (a) Find the image of each of the sets $\{a, b\}$ and $\{a, c, d\}$. (2 pts)
   (b) Find the inverse image of each of the sets $\{2\}$ and $\{0, 6\}$. (2 pts)
   (c) For the function f, determine whether it is one-to-one, and whether it is onto B. (Justify your answer). (3 pts)

Exercise 4:(4 pts)

1. Give three sets A, B and D such that: $\aleph_0 < |A| < |B| < |\mathcal{P}(D)|$, where $\aleph_0$ is the cardinal of $\mathbb{N}$. (Justify your answer). (2 pts)
2. Give the cardinal of the set A in each of the following cases. (Justify your answer).
   (a) The set $\mathbb{Z}$ of integers. (1 pts)
   (b) The set $\mathbb{Q}$ of rational numbers. (1 pts)

Answer Final examination Math 132.
Semester 1 (1442).

Exercice 1. (15 pts)

1) (3 pts)

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p \wedge \neg q) \to \neg(p \vee q)$
T	T	F	F	F	T	F	T
T	F	F	T	F	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

2) (3 pts)

$\neg(p \to q) \to q \equiv (p \to q) \vee q$

$\equiv \neg p \vee q \vee q \equiv \neg p \vee q$

$\equiv p \to q.$

3) (2 pts)

$m+n$ is even $\Rightarrow m+n = 2a;\ a \in \mathbb{Z}.$
$n+p$ is even $\Rightarrow n+p = 2b,\ b \in \mathbb{Z}.$

$\Rightarrow m+n+n+p = 2a+2b$
$\Rightarrow m+2n+p = 2a+2b$
$\Rightarrow m+p = 2a+2b-2n = 2(\overbrace{a+b-n}^{c})$
$= 2c \quad ; c \in \mathbb{Z}.$

$\Rightarrow m+p$ is even.

4) (3 pts)

$P(n): 2+4+\dots+2n = n(n+1).$
$\forall n \geq 1.$

B.S: $P(1)$: $2 = (1)(2) = 2$ true.

I.S: Let $k \geq 1$, we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$P(k): 2+4+\dots+2k = k(k+1)$ true I.H

$P(k+1): 2+4+\dots+2k+2(k+1) = (k+1)(k+2)$?

$\underbrace{2+4+\dots+2k}_{IH}+(2k+2) = k(k+1)+2(k+1)$
$= (k+1)(k+2).$

5) (4 pts)

$P(n)$: "$2 | a_n$" $\forall n \geq 0.$

B.S: $P(0): 2 | a_0 = 2$ true.
$P(1): 2 | a_1 = 6$ true.

I.S: Let $k \geq 1$; we assume that $P(0), P(1), \dots, P(k)$ are true and we prove that $P(k+1)$ is true.

$P(k+1): 2 | a_{k+1}$

$a_{k+1} = a_k + a_{k-1}$

$P(k)$ true $\Rightarrow 2 | a_k \Rightarrow a_k = 2t,\ t \in \mathbb{Z}.$

$P(k-1)$ true $\Rightarrow 2 | a_{k-1} \Rightarrow a_{k-1} = 2s;\ s \in \mathbb{Z}.$

$\Rightarrow a_{k+1} = 2t+2s \Rightarrow a_{k+1} = 2(\overbrace{t+s}^{m})$
$= 2m;\ m \in \mathbb{Z}$

$\Rightarrow a_{k+1} = 2m \Rightarrow 2 | a_{k+1}.$

Exercice 2. (11 pts)

1) (4 pts)

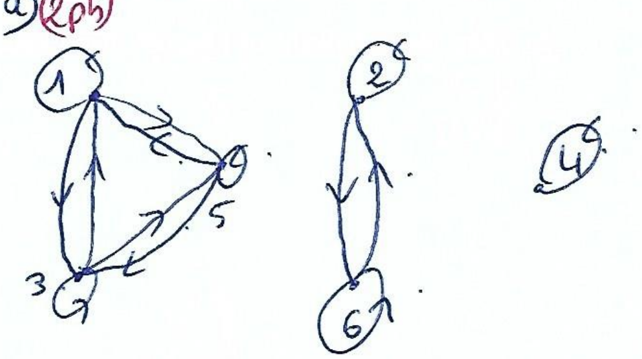
- Let $m \in \mathbb{Z}$; $m+m = 2m$ is even
  $\Rightarrow mRm \Rightarrow R$ is reflexive.
- Let $m, n \in \mathbb{Z}$; $mRn \Rightarrow m+n$ even.
  $\Rightarrow n+m$ is even.
  $\Rightarrow nRm$

$\Rightarrow R$ is symmetric and not anti-symmetric

- Let $m, n, p \in \mathbb{Z}$; mRn and nRp
  $\Rightarrow m+n$ is even and $n+p$ is even.
  $\Rightarrow m+p$ is even
  $\Rightarrow mRp \Rightarrow R$ is transitive

2) (7 pts)

a) (2 pts)



b) (3 pts)

- $(1,1)\in E$, $(2,2)\in E$, $(3,3)\in E$, $(4,4)\in E$, $(5,5)\in E$, $(6,6)\in E$

$\Rightarrow E$ is reflexive.

- $\left.\begin{array}{l}(1,3)\in E, (3,1)\in E.\\ (1,5)\in E, (5,1)\in E.\\ (2,6)\in E, (6,2)\in E.\\ (3,5)\in E, (5,3)\in E\end{array}\right\} \Rightarrow$. E is symmetric.

- $\left.\begin{array}{l}(1,3)\in E, (3,5)\in E \Rightarrow (1,5)\in E\\ (1,5)\in E, (5,3)\in E \Rightarrow (1,3)\in E.\\ \vdots\end{array}\right\} \Rightarrow E$ is transitive.

$\Rightarrow E$ is an equivalence relation on S.

c) (2 pts)

$\{1,3,5\}, \{2,6\}, \{4\}$.

Exercice 3: (10 pts)

1) (3 pts)

a) (1 pt)

$(Y \setminus X) = \phi$.

$X \times \phi = \phi$.

b) (1 pt)

$(Y \times X) = \{(2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$.

$(Y \times X) \cap Z = \{(2,2), (2,4), (2,3)\}$.

c) (1 pt)

$(Y \times X) \setminus Z = \{(2,1), (3,1), (3,2), (3,3), (3,4)\}$.

2) (7 pts)

a) (2 pts)

$f(\{a,b\}) = \{4,6\}$.

$f(\{a,c,d\}) = \{0,4\}$.

b) (2 pts)

$f^{-1}(\{2\}) = \phi$.

$f^{-1}(\{0,6\}) = \{b,c\}$.

c) (3 pts)

$f(a) = f(d)$ and $a \neq d$

$\Rightarrow f$ is not one-to-one.

. It does not exist $x \in A$ such that $f(x) = 2$, (2 haven't a preimage in A)

$\Rightarrow f$ is not onto B.

Exercice 4: (4 pts)

1) (2 pts)

. $A = P(\mathbb{N})$

because $|P(\mathbb{N})| > |\mathbb{N}| = \aleph_0$.

$B = P(P(\mathbb{N}))$ $\quad |P(P(\mathbb{N}))| > |P(\mathbb{N})|$

$B > A$.

$D = B$.

2) (2 pts)

a) (1 pt)

$P(\mathbb{Z}) = \aleph_0$ because there is a one to one correspondance from $\mathbb{Z}$ to $\mathbb{N}$.

b) (1 pt)

$P(\mathbb{Q}) = \aleph_0$.