King Saud University
Department of Mathematics

Final Exam in Math 151, S2-1446H.
(This is a two-page long exam)
Calculators are not allowed

Q1. **(a)** Without using truth tables, show that $A = [(p \wedge \neg q) \vee \neg p] \rightarrow r$ is logically equivalent to $B = \neg r \rightarrow (p \wedge q)$. (3)

(b) Use induction to prove the following for every $n \geq 1$:

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \cdots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}. \ (4)$$

(c) Let a, b and c be real numbers. Use contraposition to show that if $2a - c > 7$, then $a - b > 1$ or $c - 2b < -5$. (2)

Q2. **(a)** Let R be the relation on $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ defined by:

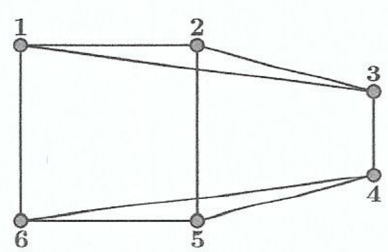
$$xRy \Leftrightarrow x - y = 10.$$

Determine whether R is reflexive, symmetric, antisymmetric or transitive. (4)

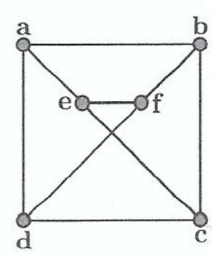
(b) Let E be the relation on $\mathbb{Z}$ defined by $mEn \Leftrightarrow 4 \mid (m^2 - n^2)$.
(i) Prove that E is an equivalence relation. (3)
(ii) Show that $[m] = [7m]$ for every $m \in \mathbb{Z}$. (1)
(iii) Show that $m + 1 \notin [m]$ for every $m \in \mathbb{Z}$. (1)

Q3. **(a)** Let G be a simple graph with degree sequence $x, x, x, x, x, x, 2x, 4x$ and having 6 edges.
(i) Find x. (1)
(ii) Show that G cannot be connected. (1)
(iii) Find the number of edges of the complement $\overline{G}$ of G. (1)

(b) Determine if the following graphs H and I are isomorphic. (2)



H



I

(c) Determine whether the graph I in **(b)** is bipartite, and if so, then find a bipartite representation. (2)

Q4.(a) (i) Give an example of a tree T, whose complement $\overline{T}$ is a tree. (1)

(ii) Give an example of a tree T, whose complement $\overline{T}$ is not a tree. (1)

(iii) Draw all (nonisomorphic) trees with 5 vertices. (2)

(b) For the graph I in **Q3(b)**, find a spanning tree with root f,

(i) using *depth-first* search; (1)

(ii) using *breadth-first* search. (1)

(c) Using alphabetical order, form a binary search tree for the words: *Lily, Daisy, Rose, Violet, Orchid, Tulip, Lotus.* (2)

Q5. **(a)** For the Boolean function $f(x, y, z) = (x + y\overline{z})(\overline{x} + y)$, find

(i) the complete sum-of-products expansion (CSP); (2)

(ii) the complete product-of-sums expansion (CPS). (2)

(b) Let $g(x, y, z) = xyz + x\overline{y}\,\overline{z} + x\overline{y}z + \overline{x}\,y\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. (1)

(ii) Simplify g (i.e., write it in MSP form). (2)

Answer Final Math151

462

9

Q1) a)

$$[(p\wedge\neg q)\vee\neg p]\to r\equiv[(p\vee\neg p)\wedge(\neg q\vee\neg p)]\to r$$

$$\equiv(T\wedge(\neg q\vee\neg p))\to r$$

3

$$\equiv(\neg q\vee\neg p)\to r$$

$$\equiv\neg r\to\neg(\neg q\vee\neg p)$$

$$\equiv\neg r\to(p\wedge q)$$

b) $P(n):\ \dfrac{3}{1^2\times2^2}+\dfrac{5}{2^2\times3^2}+\cdots+\dfrac{2n+1}{n^2(n+1)^2}=1-\dfrac{1}{(n+1)^2}\quad,\ \forall n\geq1.$

B.S: $P(1)$: $\dfrac{3}{1^2\times2^2}=\dfrac{3}{4}$. L.H.S.

$1-\dfrac{1}{2^2}=\dfrac{3}{4}$ R.H.S

$\Rightarrow P(1)$ is true.

I.S: Let $k\geq1$; we assume that $P(k)$ is true and we show that $P(k+1)$ is true.

$P(k)$: $\dfrac{3}{1^2\times2^2}+\dfrac{5}{2^2\times3^2}+\cdots+\dfrac{2k+1}{k^2(k+1)^2}=1-\dfrac{1}{(k+1)^2}.$

$P(k+1)$: $\dfrac{3}{1^2\times2^2}+\dfrac{5}{2^2\times3^2}+\cdots+\dfrac{2k+3}{(k+1)^2(k+2)^2}=1-\dfrac{1}{(k+2)^2}$?

$$\frac{1}{1^2\times2^2}+\frac{5}{2^2\times3^2}+\cdots+\frac{2k+1}{(k+1)^2(k+2)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}.$$

4

$$=1-\frac{1}{(k+1)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}$$

$$=1-\frac{(k+2)^2-(2k+3)}{(k+1)^2(k+2)^2}=1-\frac{k^2+4k+4-2k-3}{(k+1)^2(k+2)^2}$$

$$=1-\frac{(k+1)^2}{(k+1)^2(k+2)^2}=1-\frac{1}{(k+2)^2}$$

$\Rightarrow P(k+1)$ is true $\Rightarrow\forall n\geq1$ $P(n)$ true.

c) By contraposition, we prove that: if $a-b\leq 1$ and $c-2b\geqslant -5$ then $2a-c\leq 7$.

we assume that $a-b\leq 1$ and $c-2b\geqslant -5$.

$\Rightarrow 2a-2b\leq 2$ and $2b-c\leq 5$.

2

$\Rightarrow 2a-2b+2b-c\leq 7$.

$\Rightarrow 2a-c\leq 7$.

9

Q2) a) $\cdot\ x\in\mathbb{Z}^+;\ x-x=0\neq 10$

$\Rightarrow x \not R x \Rightarrow R$ is not reflexive.

$\cdot\ x,y\in\mathbb{Z}^+;\ xRy \Rightarrow x-y=10 \Rightarrow y-x=-10\neq 10$

$\Rightarrow y \not R x$

$\Rightarrow R$ is not symmetric.

$\cdot\ x,y\in\mathbb{Z}^+;\ xRy$

$x\neq y$.

4

$\Rightarrow x-y=10$.

$\Rightarrow y-x=-10\neq 10$

$\Rightarrow y\not R x \Rightarrow R$ is antisymmetric.

$\cdot\ 10,20,30\in\mathbb{Z}^+;\ 30\,R\,20$

$20\,R\,10$

by $30\not R\,10 \Rightarrow R$ is not transitive

b) i) $\cdot\ m\in\mathbb{Z};\ m^2-m^2=0\Rightarrow 4|0\Rightarrow 4|(m^2-m^2)\Rightarrow mEm$.

$\Rightarrow E$ is reflexive.

$\cdot\ m,n\in\mathbb{Z};\ mEn\Rightarrow 4|(m^2-n^2)\Rightarrow 4|(n^2-m^2)$

3

$\Rightarrow nEm \Rightarrow E$ is symmetric

$\cdot\ m,n,p\in\mathbb{Z};\ mEn$ and $nEp\Rightarrow 4|(m^2-n^2)$ and $4|(n^2-p^2)$

$\Rightarrow 4|(m^2-n^2+n^2-p^2)\Rightarrow 4|(m^2-p^2)$

$\Rightarrow mEp \Rightarrow E$ is transitive

ii) $(7m)^2 - m^2 = 49m^2 - m^2 = 48m^2$
$\Rightarrow 4 \mid 48m^2$
$\Rightarrow (7m)\, E\, m \Rightarrow [m] = [7m]$ 1

iii) $(m+1)^2 - m^2 = 2m+1$; $4 \nmid (2m+1) \Rightarrow (m+1) \not E\, m$.
$\Rightarrow (m+1) \notin [m]$. 1

Q3) 7 a) i) $12x = 12 \Rightarrow x = 1$. 1

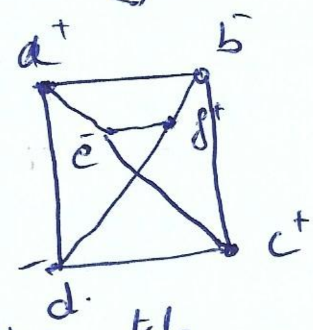
ii) A connected graph have minimum $(n-1)$ edges. $6 \ngtr 7$. 1

iii)

$|E(G)| = \frac{8 \times 7}{2} - 6 = 22$. 1

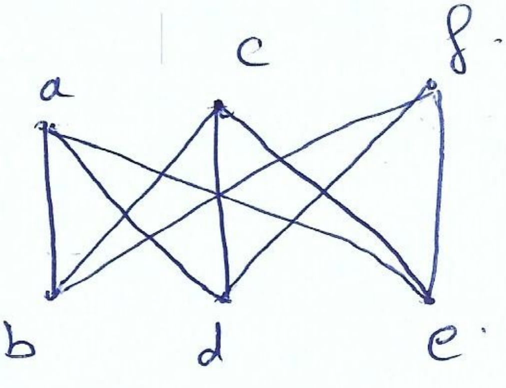
b) No, because H contains 2 simple circuits of length 3 but I doesnot contain any circuits of length 3. 2

c)



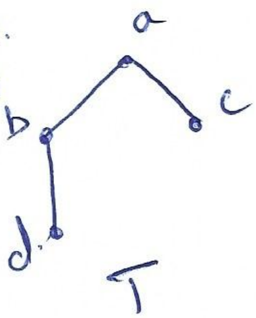
yes bipartite

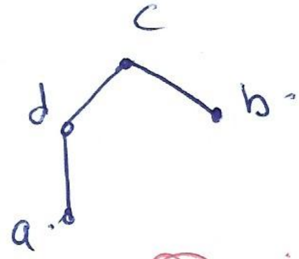
2



Q4) 8

a) i)





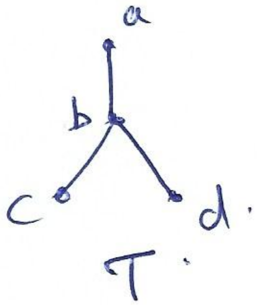
1

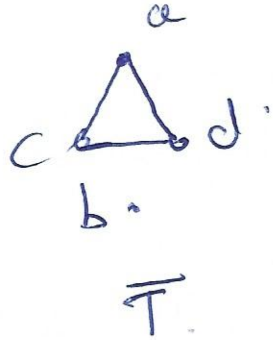
$(n-1)+(n-1)=\frac{n(n-1)}{2}$.

$2(n-1)=\frac{n}{2}(n-1)$

$\rightarrow n=4$.

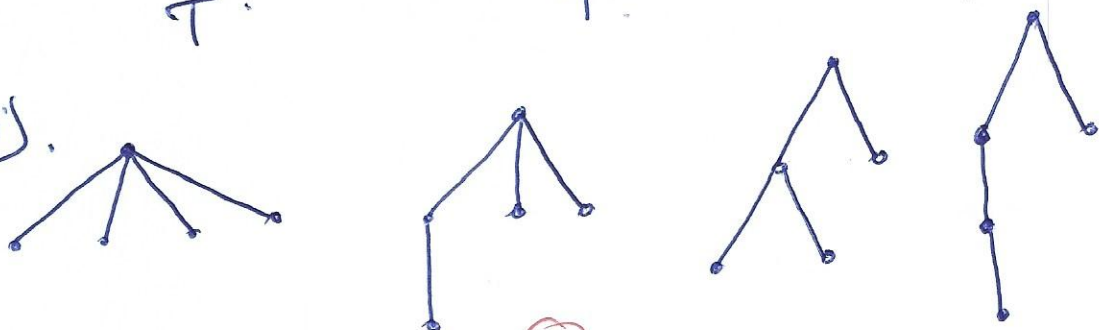
ii)





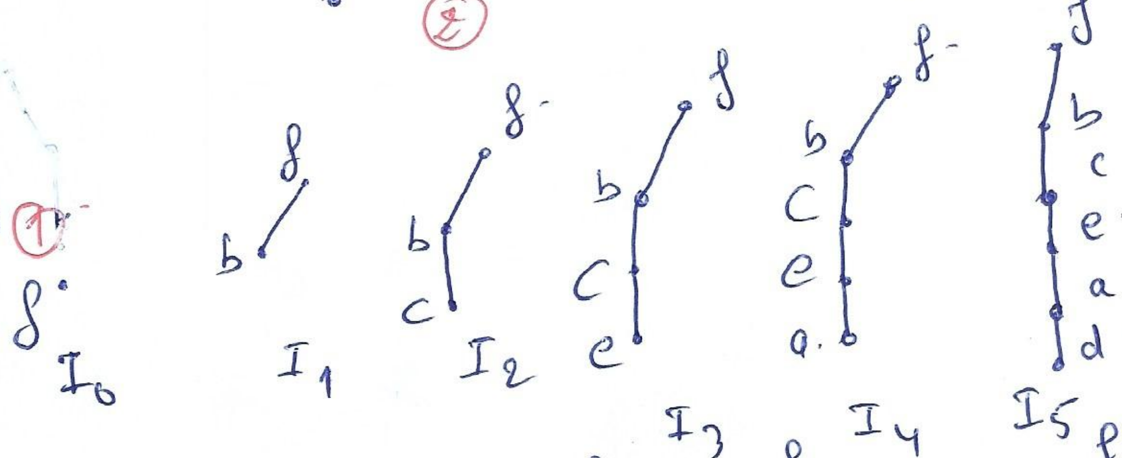
1

iii).

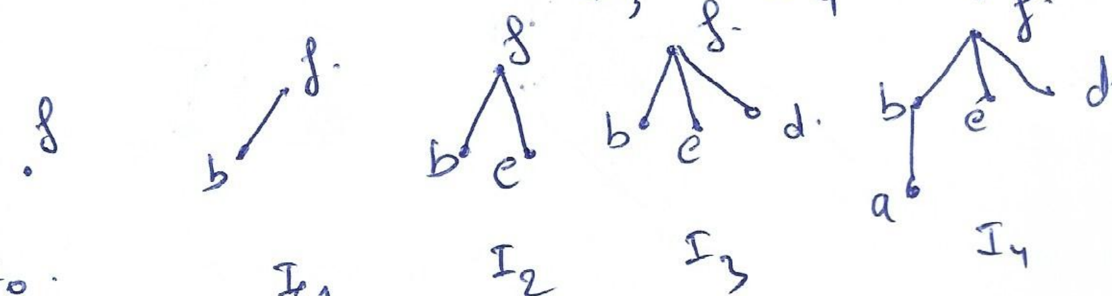


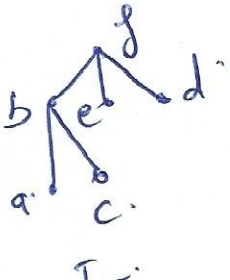
2

b) i) 1



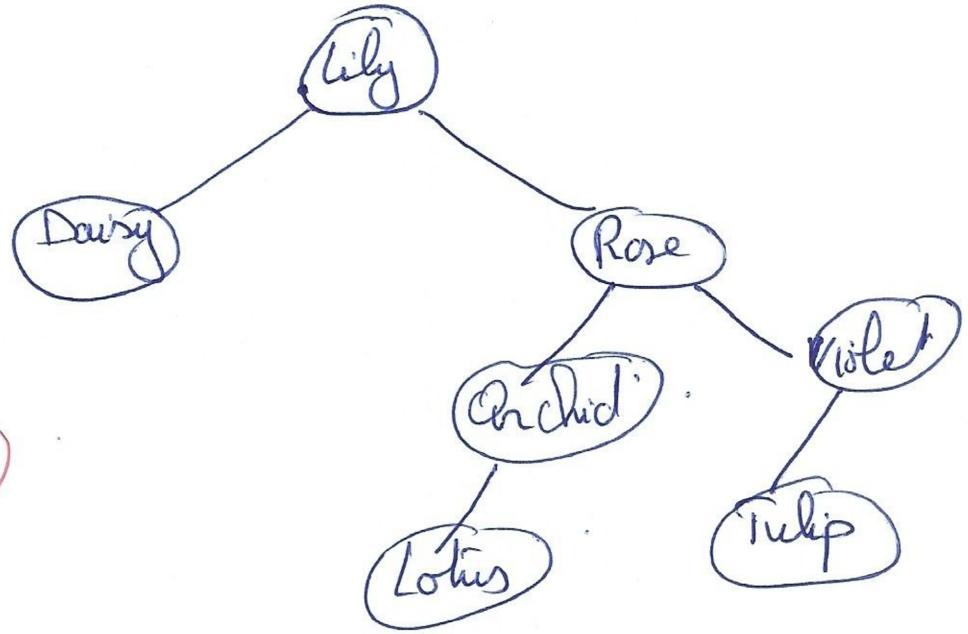
b) ii)





4

c)



2

Q5) 7

a) i)

$$f(x,y,z) = (x + y\bar{z})(\bar{x} + y)$$
$$= x\bar{x} + xy + \bar{x}y\bar{z} + y\bar{z}$$
$$= xy + \bar{x}y\bar{z} + y\bar{z}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$		1		

2

$$CSP(f) = xyz + xy\bar{z} + \bar{x}y\bar{z}$$

ii) $CSP(\bar{f}) = x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$

$\Rightarrow CPS(f) = (\bar{x}+y+z)(\bar{x}+y+\bar{z})(x+\bar{y}+\bar{z})(x+y+z)(x+y+\bar{z})$

b) i)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1		1	1
$\bar{x}$		1		1

1 2

ii) $MSP(g) = xz + x\bar{y} + \bar{y}z + \bar{x}y\bar{z}$

2