King Saud University	College of Sciences		Department of Mathematics
First Examination	Math 228	1446(Semester 1)	Time:1H30

Name:······································· Student Number:··············· Grade: ··· /25

Question 1: (8 marks)

1. Find the average value of the function $5x - 7$ on the interval $[-1, 1]$. **(2 marks)**

$$f_{av} = \frac{1}{1-(-1)} \int_{-1}^{1} (5x-7)\, dx.$$

$$= \frac{1}{2}\left[\frac{5x^2}{2} - 7x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\cancel{\frac{5}{2}} - 7\right) - \left(\cancel{\frac{5}{2}} + 7\right)\right].$$

$$= \frac{1}{2} \times (-14) = -7.$$

2. Find the derivative of $F(x) = \displaystyle\int_{2x}^{x^2} \frac{1}{\ln t^3 + 1}\, dt$ **(2 marks)**

$$F'(x) = \frac{1}{\ln(x^2)^3 + 1} \cdot 2x - \frac{1}{\ln(2x)^3 + 1} \cdot 2.$$

$$= \frac{2x}{\ln(x^6) + 1} - \frac{2}{\ln(8x^3) + 1}.$$

3. Find the area enclosed between the graphs $y = x$ and $y = x^2 - 2$. **(2 marks)**

$y = x\,,\ y = x^2 - 2 \Rightarrow x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0$.

$\Rightarrow x = -1$ or $x = 2$.

x		-1		2	
$x+1$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
x^2-x-2	$+$	0	$-$	0	$+$

$\Rightarrow x \geq x^2 - 2$ in $[-1, 2]$

$$A = \int_{-1}^{2} (x - x^2 + 2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} = \left[2 - \frac{8}{3} + 4\right] - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$$

$$= 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}.$$

4. Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis. (Use Disk method) **(2 marks)**

$-x^2 + x = 0 \Rightarrow x(-x+1) = 0 \Rightarrow x = 0$ or $x = 1$.

$$V = \pi \int_0^1 (-x^2 + x)^2\,dx = \pi \int_0^1 (x^4 - 2x^3 + x^2]\,dx.$$

$$= \pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \pi \left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \pi \left[\frac{6 - 15 + 10}{30}\right]$$

$$= \frac{\pi}{30}$$

Question 2: (17 marks) Evaluate the following integrals:

1. $\int \frac{1}{\cos^2(\pi x)}\, dx$. **(2 marks)**

$u = \pi x \Rightarrow du = \pi\, dx.$

$= \frac{1}{\pi}\int \frac{1}{\cos^2 u}\, du = \frac{1}{\pi}\int \sec^2 u\, du.$

$= \frac{1}{\pi} \tan(u) + C.$

$= \frac{1}{\pi} \tan(\pi x) + C.$

2. $\int x e^x\, dx$. **(2 marks)**

$u = x \rightarrow du = dx.$

$dv = e^x dx \rightarrow v = e^x.$

$= xe^x - \int e^x\, dx.$

$= xe^x - e^x + C.$

3. $\int \sinh^5 x \cosh^4 x \, dx$. **(3 marks)**

$u = \cosh x \longrightarrow du = \sinh x \, dx.$

$\sinh^2 x = \cosh^2 x - 1.$

$= \int (\sinh^2 x)^2 \cosh^4 x \cdot \sinh x \, dx.$

$= \int (\cosh^2 x - 1)^2 \cosh^4 x \sinh x \, dx$

$= \int (u^2 - 1)^2 u^4 \, du.$

$= \int (u^4 - 2u^2 + 1) u^4 \, du = \int (u^8 - 2u^6 + u^4) \, du$

$= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C = \frac{\cosh^9 x}{9} - \frac{2\cosh^7 x}{7} + \frac{\cosh^5 x}{5} + C.$

4. $\int \frac{x^2}{\sqrt{9 - x^2}} \, dx$. **(3 marks)**

$x = 3\sin\theta \Rightarrow dx = 3\cos\theta \, d\theta.$

$= \int \frac{9\sin^2\theta}{\sqrt{9 - 9\sin^2\theta}} 3\cos\theta \, d\theta.$

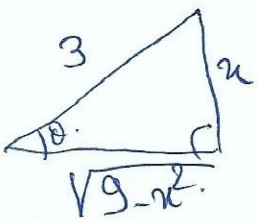
$= \int \frac{9\sin^2\theta}{\sqrt{3^2(1 - \sin^2\theta)}} \cancel{3}\cos\theta \, d\theta = \int 9\sin^2\theta \, d\theta.$

$\theta = \sin^{-1}\left(\frac{x}{3}\right).$

$= 9\int \frac{1 - \cos 2\theta}{2} \, d\theta.$

$\sin\theta\cos\theta.$

$= \frac{9}{2}\theta - \frac{1}{2}\sin 2\theta + C.$



$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2} 2 \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9 - x^2}}{9} + C.$

5. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}dx$ **(2 marks)**

$$= \int \frac{\frac{1}{x}}{\sqrt{1^2-(\ln(x))^2}}\,dx = \sin^{-1}(\ln(x)) + C.$$

6. $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$ **(3 marks)**

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)^2} + \frac{B}{(x+2)}.$$

$$= \frac{A + B(x+2)}{(x+2)^2}.$$

$$\Rightarrow 6x+7 = Bx + 2B + A. \quad \Rightarrow B = 6.$$

$$2B+A = 7. \Rightarrow A = 7-12 = -5.$$

$$\int \frac{6x+7}{(x+2)^2}\,dx = -\int \frac{5}{(x+2)^2}\,dx + \int \frac{6}{x+2}\,dx$$

$$= -5\int (x+2)^{-2}\,dx + 6\int \frac{1}{x+2}\,dx$$

$$= -5\,\frac{(x+2)^{-1}}{-1} + 6\ln|x+2| + C$$

$$= \frac{5}{x+2} + 6\ln|x+2| + C.$$

7. $\int \frac{1}{2+\cos x}\,dx$. **(2 marks)**

$u = \tan\frac{x}{2} \quad dx = \frac{2}{u^2+1}\,du$

$\cos x = \frac{1-u^2}{1+u^2}$.

$$\int \frac{1}{2+\cos x}\,dx = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{u^2+1}\,du$$

$$= \int \frac{2}{2+2u^2+1-u^2}\,du$$

$$= \int \frac{2}{u^2+3}\,du$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C.$$