King Saud University	College of Sciences	Department of Mathematics	
First Examination	Math 132	Semester II (1442)	Time:1H30

Exercise 1: (11 pts)

1. Decide whether the following proposition is a tautology or a contradiction or a contingency?
$$(p \to q) \vee (\neg q \vee r). \quad \text{(3 pts)}$$
2. Without using truth tables, prove that the following conditional statement is a Tautology:
$$q \to [p \vee (\neg p \wedge q)]. \quad \text{(3 pts)}$$
3. Without using truth tables, prove the following logical equivalence:
$$p \wedge [p \vee (\neg p \wedge q)] \equiv p. \quad \text{(3 pts)}$$
4. Determine the truth value of each of the following statements if the domain consists of all real numbers. (Justify your answer)
   (a) $\exists x \in \mathbb{R};\ (x^4 < x^3)$. (1 pts)
   (b) $\forall x \in \mathbb{R};\ (-x)^3 = -x^3$. (1 pts).

Exercise 2: (14 pts)

1. Let n be an integer. Prove that: $2n^2 + n$ is odd if and only if n is odd. (3 pts)
2. Let x, y and z be three real numbers. Prove that:
   if $(3x^2 + y^3 + 5z = 64)$ then, $(x \geq 2$ or $y \geq 3$ or $z \geq 5)$. (2 pts)
3. (a) Using a proof by contradiction, prove that: if x is an irrational number and y is a rational number, then $x + y$ is an irrational number. (2 pts)
   (b) Deduce that $\dfrac{3}{2} + \sqrt{3}$ is an irrational number. (1 pts)
4. Use mathematical induction to prove the following statement:
   $3+3\times 2^1+3\times 2^2+\cdots+3\times 2^n = 3(2^{n+1}-1)$, for each integer n, with $n \geq 0$. (3 pts)
5. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_0 = 0 \\ u_1 = 4 \\ u_{n+1} = 6u_n - 5u_{n-1}; \quad n \geq 1 \end{cases}$
   Use mathematical induction to prove the following statement:
$$u_n = 5^n - 1, \quad \text{for each integer } n, \text{ with } n \geq 0. \quad \text{(3 pts)}$$

Answer mid 1 (1442 (II)).

Exercise 1: (10pts)

1) (3pts)

p	q	r	$p\to q$	$\neg q$	$\neg q\vee r$	$(p\to q)\vee(\neg q\vee r)$
T	T	T	T	F	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

is a tautology.

2) (3pts)

$q\to[p\vee(\neg p\wedge q)]$

$\equiv q\to[(p\vee\neg q)\wedge(p\vee q)]$.

$\equiv q\to[T\wedge(p\vee q)]$

$\equiv \neg q\vee(p\vee q)$

$\equiv p\vee(q\vee\neg q)\equiv p\vee T\equiv T$.

3) (3pts)

$p\wedge[p\vee(\neg p\wedge q)]$

$\equiv [p\ (\neg p\wedge q)] \equiv p\wedge[(p\vee\neg p)\wedge(p\vee q)]$

$\equiv p\wedge[T\wedge(p\vee q)]$

$\equiv p\wedge(p\vee q)$

$\equiv p$.

4)

(1pts) a) True, $x=\frac{1}{2}$; $\left(\frac{1}{2}\right)^4<\left(\frac{1}{2}\right)^3$

(1pts) b) True.

$(-x)^3=(-1)^3.x^3=-x^3$; $\forall x\in\mathbb{R}$

Exercise 2: (15pts)

1) (3pts)

• if $2n^2+n$ is odd then n is odd.

by contraposition, if n is even then $2n^2+n$ even

$n=2k\Rightarrow 2n^2+n=(2k)^2+2k$
$=2(2k^2+k)$ even.

• if n is odd, then $2n^2+n$ is odd.

$n=2k+1$;

$2n^2+n=2(2k+1)^2+2k+1$
$=2(4k^2+4k+1)+2k+1$
$=2(4k^2+4k+1+k)+1$ is odd.

2) (2pts) By contraposition, we prove that:

if $x<2$ and $y<3$ and $z<5$

then $3x^2+y^3+5z\neq 64$.

$x<2\Rightarrow 3x^2<12$
$y<3\Rightarrow y^3<27$
$z<5\Rightarrow 5z<25$
$\Rightarrow 3x^2+y^3+5z<64$
$\Rightarrow 3x^2+y^3+5z\neq 64$.

3) (2pts)

a) by contradiction, we assume that $x+y$ is rational;

so $x+y=\frac{a}{b}$; $a,b\in\mathbb{Z}$; $b\neq 0$.

y is rational, so $y=\frac{c}{d}$; $c,d\in\mathbb{Z}$; $d\neq 0$

$\Rightarrow x=\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}=\frac{m}{n}$.

$m=ad-bc\in\mathbb{Z}$, $n=bd\neq 0$
$\in\mathbb{Z}$

$\Rightarrow x$ is rational, a contradiction.

b) (1pts)

we take $x=\sqrt{3}$ and $y=\frac{3}{2}$

we have the result from (a).

4) (3pts)

$P(n): 3+3\times 2^1+\dots+3\times 2^n = 3(2^{n+1}-1)$.

$n \geq 0$.

B.S: $P(0): 3 = 3(2-1) = 3$. True.

I.S: Let $k \geq 0$, we prove $P(k) \to P(k+1)$.

we assume that $P(k)$ is true.

$3+3\times 2^1+\dots+3\times 2^k = 3(2^{k+1}-1)$ IH

and we prove that $P(k+1)$ is true.

$3+3\times 2^1+\dots+3\times 2^{k+1} = 3(2^{k+2}-1)$?

$\underbrace{3+3\times 2^1+\dots+3\times 2^k}_{IH}+3\times 2^{k+1} =$

$= 3(2^{k+1}-1)+3\times 2^{k+1}$

$= 3(2^{k+1}+2^{k+1}-1)$

$= 3(2\times 2^{k+1}-1) = 3(2^{k+2}-1)$

So $P(k+1)$ is true

$\Rightarrow \forall n \geq 0$ $P(n)$ true.

5) (4pts)

$P(n): u_n = 5^n-1$. $\forall n \geq 0$.

B.S: $P(0): u_0 = 5^0-1 = 0$ true.

$P(1) = 5^1-1 = 4$ true.

I.S: Let $k \geq 1$, we assume that $P(0), P(1), \dots, P(k)$ are true

and we prove that $P(k+1)$ is true:

$P(k+1): u_{k+1} = 5^{k+1}-1$?

$u_{k+1} = 6u_k - 5u_{k-1}$

$P(k)$ true $\Rightarrow u_k = 5^k-1$

$P(k-1)$ true $\Rightarrow u_{k-1} = 5^{k-1}-1$

$u_{k+1} = 6(5^k-1)-5(5^{k-1}-1)$

$= 6\times 5^k-6-5\times 5^{k-1}+5$

$= 6\times 5^k-5^k-1$

$= 5^k(6-1)-1$

$= 5^k\times 5-1$

$= 5^{k+1}-1$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 0$ $P(n)$ true.