King Saud University	College of Sciences	Department of Mathematics
Final Examination	Math 132 Semester I(1447)	Time: 3H

Calculators are not allowed

The Examination contain 2 pages

Question 1: (12 marks)

1. Without using truth tables, prove the following logical equivalence: **(3 marks)**
$$\neg[p \vee (\neg p \wedge q)] \equiv (\neg p \wedge \neg q)$$
2. Without using truth tables, prove that the following statement is a contradiction: **(3 marks)**
$$p \wedge (p \to q) \wedge (p \to \neg q).$$
3. Given real numbers x, y, and z, prove by contradiction the following statement: if $(x + y + z = 58)$, then $(x \geq 16$ or $y \geq 38$ or $z \geq 4)$. **(2 marks)**
4. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined as follows:
$a_0 = 1, a_1 = 2$, and $a_{n+1} = 5a_n - 6a_{n-1}; \forall n \geq 1$.
Use mathematical induction to prove that $a_n = 2^n$, for each integer n, with $n \geq 0$. **(4 marks)**

Question 2: (12 marks)

1. Consider the relation P on the set $A := \{a, b, c, d, e\}$ defined by:
$P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (c, b), (c, d), (d, e), (c, e)\}$.
   (a) Represent P with a digraph. **(1 mark)**
   (b) Show that P is a partial ordering relation. **(3 marks)**
   (c) Is P a total ordering? (Justify your answer). **(1 mark)**
   (d) Draw the Hasse diagram of P. **(2 marks)**
2. Let E be the relation defined on the set of real numbers as follows:
$$(a\ E\ b) \text{ if and only if } (a - b) \text{ is an integer.}$$
   (a) Prove that E is an equivalence relation. **(3 marks)**
   (b) What is the equivalence class $[1]$, of the element 1 for this equivalence relation? **(1 mark)**
   (c) Is $\frac{3}{2} \in [\frac{2}{3}]$? **(1 mark)**

Question 3: (10 marks)

1. Consider the following three sets $X := \{1,2,3,4\}, Y := \{2,3\}$, and $Z := \{(1,2),(1,4),(2,2),(2,4),(4,4),(2,3)\}$. Find the following sets:
   (a) $X \times (Y \setminus X)$. **(1 mark)**
   (b) $(Y \times X) \cap Z$. **(1 mark)**
   (c) $(Y \times X) \setminus Z$. **(1 mark)**
   (d) $\{\emptyset\} \times X$. **(1 mark)**

2. Consider the sets $A := \{a,b,c,d,e\}$ and $B := \{1,2,3,4\}$, and the function $f : A \longrightarrow B$ defined by: $f(a) = 1, f(b) = f(c) = 2$, and $f(d) = f(e) = 3$.
   (a) Find the image of each of the sets $\{a,b,c\}, \{d,e\}$ and $\{b,c,d,e\}$. **(2 marks)**
   (b) Find the inverse image of each of the sets $\{1,2,3\}, \{3\}$ and $\{4\}$. **(2 marks)**
   (c) For the function f, determine whether it is one-to-one, and whether it is onto B. (Justify your answer). **(2 marks)**

Question 4: (6 marks)

1. Give an example of two uncountable sets A and B, such that $A - B$ is infinite and countable. **(2 marks)**
2. Give the cardinal of the set A in each of the following cases. (Justify your answer).
   (a) The set $\mathbb{Q}^+$ of rational numbers. **(1 mark)**
   (b) The set $A_1 := [0,1] \cap \mathbb{Z}^+$. **(1 mark)**
   (c) The set $A_2 := \mathbb{Q}^+ \cup \mathbb{Z}^+$. **(1 mark)**
   (d) The set $A_3 := \{k \in \mathbb{Z}^+; k \text{ is odd and } k \leq 20\} \cap \mathbb{R}$. **(1 mark)**

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Q1) 12

1) $\neg[p\vee(\neg p\wedge q)] \equiv \neg((p\vee\neg p)\wedge(p\vee q)]$

3

$\equiv \neg(T\wedge(p\vee q))$

$\equiv \neg(p\vee q)\equiv \neg p\wedge\neg q.$

2)

$p\wedge(p\longrightarrow q)\wedge(p\longrightarrow\neg q)$

$\equiv p\wedge[p\longrightarrow(q\wedge\neg q)] \equiv p\wedge[p\longrightarrow F]$

$\equiv p\wedge[\neg p\vee F]$

3

$\equiv p\wedge\neg p \equiv F.$

3) By contradiction, suppose that: $x<16$ and $y<38$ and $z<4$

$\Rightarrow x+y+z<16+38+4 \Rightarrow x+y+z<58$

$\Rightarrow 58<58$ a contradiction

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$\Rightarrow x\geq 16$, or $y\geq 38$ or $z\geq 4$.

4) $P(n): a_n=2^n;\ \forall n\geq 0.$

B.S.: $P(0): a_0=2^0=1$ True.

$P(1): a_1=2^1=2$ True.

I.S.: let $k\geq 1$; assume that $P(0), P(1),\dots,P(k)$ are true and we prove that $P(k+1)$ is true.

4

$P(k+1): a_{k+1}=2^{k+1}.$

$a_{k+1}=5a_k-6a_{k-1}.$ $\qquad P(k)$ true $\Rightarrow a_k=2^k.$

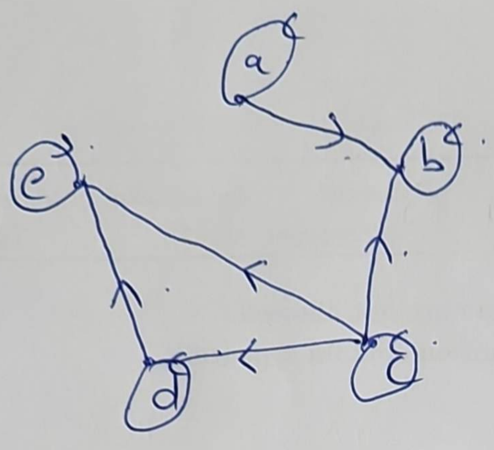
$=5\times 2^k-6\times 2^{k-1}$ $\qquad P(k-1)$ true $\Rightarrow a_{k-1}=2^{k-1}$

$=5\times 2^k-3\times 2^k.$

$=2\times 2^k=2^{k+1}$ $\Rightarrow P(k+1)$ is true $\Rightarrow P(n)$ is true

$Q_2)$ 12

1) a)



1

b) • $(a,a)\in P; (b,b)\in P, (c,c)\in P, (d,d)\in P, (e,e)\in P \Rightarrow$ P is reflexive.

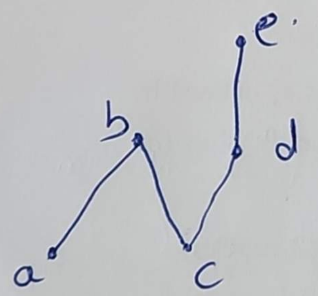
• $(a,b)\in P, (b,a)\notin P; (c,b)\in P; (b,c)\notin P; (c,d)\in P, (d,c)\notin P;$

3 $(c,e)\in P; (e,c)\notin P; (d,e)\in P; (e,d)\notin P \Rightarrow$ P is antisymmetric.

• $\left.\begin{matrix}(c,d)\in P\\(d,e)\in P\end{matrix}\right\}$ and $(c,e)\in P \Rightarrow$ P is transitive.

c) 1 No, P is not a total order, a, c are incomparables.

d)



2

2) a) • $a\in\mathbb{R};\ a-a=0\in\mathbb{Z} \Rightarrow aEa \Rightarrow$ E is reflexive

• $a,b\in\mathbb{R};\ aEb \Rightarrow (a-b)\in\mathbb{Z} \Rightarrow (b-a)\in\mathbb{Z} \Rightarrow bEa \Rightarrow$ E is symmetric

3 • $a,b,c\in\mathbb{R};\ aEb$ and $bEc \Rightarrow (a-b)\in\mathbb{Z}$ and $(b-c)\in\mathbb{Z}$.

$\Rightarrow [(a-b)+(b-c)]\in\mathbb{Z}$.

$\Rightarrow (a-c)\in\mathbb{Z} \Rightarrow aEc \Rightarrow$ E is transitive

$\Rightarrow$ E is an equivalence relation.

b) $[1]=\{b\in\mathbb{Z};\ bE1\}=\{b\in\mathbb{Z};\ (b-1)\in\mathbb{Z}\}=\mathbb{Z}$ 1

c) $\frac{3}{2}-\frac{2}{3}=\frac{9-4}{6}=\frac{5}{6}\notin\mathbb{Z} \Rightarrow \frac{3}{2}\notin\left[\frac{2}{3}\right]$ 1

$Q_3)$ 10

1) a) $(Y \setminus X) = \phi \Rightarrow X \times (Y \setminus X) = \phi$ 1

b) $(Y \times X) = \{(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$.

$(Y \times X) \cap Z = \{(2,2),(2,4),(2,3)\}$. 1

c) $(Y \times X) \setminus Z = \{(2,1),(3,1),(3,2),(3,3),(3,4)\}$. 1

d) $\{\phi\} \times X = \{(\phi,1),(\phi,2),(\phi,3),(\phi,4)\}$. 1

2) a) $f(\{a,b,c\}) = \{1,2\}$; $f(\{d,e\}) = \{3\}$; $f(\{b,c,d,e\}) = \{2,3\}$. 2

b) $f^{-1}(\{1,2,3\}) = A = \{a,b,c,d,e\}$; $f^{-1}(\{3\}) = \{d,e\}$.

$f^{-1}(\{4\}) = \phi$. 2

c) $f(b) = f(c) \Rightarrow f$ is not one to one. 1

$f^{-1}(\{4\}) = \phi \Rightarrow f$ is not onto. 1

$Q_4)$ 10

1). $A = \mathbb{R}$, $B = \mathbb{R} - \mathbb{Z}^+$.

$\mathbb{R}$ is uncountable, $\mathbb{R} - \mathbb{Z}^+$ is uncountable.

$A - B = \mathbb{Z}^+$ countable. 2

2). a) $|\mathbb{Q}^+| = \aleph_0$. 1

b) $A_1 = [0,1] \cap \mathbb{Z}^+ = \{1\}$.

$|A_1| = 1$. 1

c). $A_2 = \mathbb{Q}^+ \cup \mathbb{Z}^+$; $|A_2| = \aleph_0$. $(|\mathbb{Q}^+| = \aleph_0; |\mathbb{Z}^+| = \aleph_0)$. 1

d) $A_3 = \{1,3,5,7,9,11,13,15,17,19\}$, $|A_3| = 10$. 1

3)