King Saud University	College of Sciences	Department of Mathematics	
Final Examination	Math 228	Semester 1 (1446)	Time: 3 Hours

Question I: (14 marks)

1. Find the volume V of the solid in the first octant ($x \geq 0$, $y \geq 0$) bounded by the coordinates planes and the graphs of equations $z = x^2 + y + 1$ and $2x + y = 2$. **(3 marks)**

2. Use polar coordinates to evaluate the integral $\displaystyle\iint_R (x^2+y^2)^{\frac{3}{2}}\, dA$, where R is the region bounded by the graph $x^2 + y^2 = 4$. **(3 marks)**

3. Evaluate the integral $\displaystyle\iiint_Q (x + 2y + 4z)\, dV$,

   where $Q = \{(x, y, z) : 1 \leq x \leq 2,\ -1 \leq y \leq 0,\ 0 \leq z \leq 3\}$. **(2 marks)**

4. Use cylindrical coordinates to evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-\sqrt{x^2+y^2}} z\, dz\, dy\, dx$. **(3 marks)**

5. Evaluate the integral $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z\, dz\, dy\, dx$ by changing to spherical coordinates. **(3 marks)**

Question II: (17 marks)

1. Find the partial sum S_n of the geometric sequence that satisfies the given conditions, $a = -3$, $r = \frac{1}{2}$ and $n = 9$. **(2 marks)**

2. Find the limit of the sequence $\left\{ \dfrac{5n}{e^{2n}} \right\}$. **(2 marks)**

3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3 - 6}{2n^3 - 6}$ converges or diverges. **(2 marks)**

4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{3}{4^n} \right)$ converges or diverges, and if it converges find its sum. **(3 marks)**

5. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. (Hint: use the integral test) **(2 marks)**

6. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n+n^2}{n^3+1}$ converges or diverges. **(3 marks)**

7. Determine whether the series $\sum_{n=1}^{\infty} \frac{2^{n-1}}{5^n(n+1)}$ is absolutely convergent, conditionally convergent, or divergent. **(3 marks)**

Question III: (9 marks)

1. Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{n}$. **(4 marks)**

2. Find the power series representation of the function $\frac{x}{1-x^2}$ where $|x| < 1$, and then use this result to find the power series representation of the function $\ln(1-x^2)$. **(3 marks)**

3. Find the power series representation of $f(x) = x\tan^{-1}(x^2)$. **(3 2 marks)**

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1)

$2x+y=2 \Rightarrow y=2-2x.$

$y=0 \Rightarrow 2-2x=0$

$\Rightarrow 2=2x \Rightarrow x=1.$

$\Rightarrow 0\leq x\leq 1;\ 0\leq y\leq 2-2x.$

$$V=\int_0^1\int_0^{2-2x}(x^2+y+1)\,dy\,dx$$

$$=\int_0^1\left(x^2y+\frac{y^2}{2}+y\right)_0^{2-2x}dx.$$

$$=\int_0^1\left(x^2(2-2x)+\frac{(2-2x)^2}{2}+2-2x\right)dx \quad 3$$

$$=\int_0^1\left(2x^2-2x^3+2(x^2-2x+1)+2-2x\right)dx.$$

$$=\int_0^1(-2x^3+4x^2-6x+4)\,dx=\left[-\frac{x^4}{2}+\frac{4x^3}{3}-3x^2+4x\right]_0^1$$

$$=-\frac{1}{2}+\frac{4}{3}-3+4$$

$$=\frac{-3+8+6}{6}=\frac{11}{6}.$$

2)

$x^2+y^2=4 \Rightarrow x=r\cos\theta,\ y=r\sin\theta.$

$0\leq r\leq 2,\ 0\leq\theta\leq 2\pi.\quad x^2+y^2=r^2$

$$\Rightarrow \iint_R (x^2+y^2)^{3/2}\,dA=\int_0^{2\pi}\int_0^2 (r^2)^{3/2}\,r\,dr\,d\theta.$$

$$=\int_0^{2\pi}d\theta\int_0^2(r^4)\,dr=(\theta)_0^{2\pi}\left(\frac{r^5}{5}\right)_0^2 \quad 3.$$

$$=2\pi\cdot\frac{32}{5}=\frac{64\pi}{5}.$$

3)

$$\iiint_Q (x+2y+4z)\,dV = \int_1^2\int_{-1}^0\int_0^3 (x+2y+4z)\,dz\,dy\,dx.$$

$$= \int_1^2\int_{-1}^0 \left[xz+2yz+2z^2\right]_0^3$$

$$= \int_1^2\int_{-1}^0 (3x+6y+18)\,dy\,dx.$$

$$= \int_1^2 (3xy+3y^2+18y)_{-1}^0\,dx.$$

$$= \int_1^2 (0-(-3x+3-18))\,dx$$

$$= \int_1^2 (3x+15)\,dx = \left[\frac{3x^2}{2}+15x\right]_1^2$$

$$= \left(\frac{3\times4}{2}+15\times2\right)-\left(\frac{3}{2}+15\right) \quad 2$$

$$= \frac{9}{2}+15 = \frac{39}{2}.$$

4). $0\leq x\leq 1$; $0\leq y\leq\sqrt{1-x^2}$, $0\leq z\leq 2-\sqrt{x^2+y^2}$.

$0\leq\theta\leq\frac{\pi}{2}$; $0\leq r\leq 1$; $0\leq z\leq 2-r$.

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-\sqrt{x^2+y^2}} z\,dz\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{2-r} zr\,dz\,dr\,d\theta.$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\left[\frac{z^2}{2}\right]_0^{2-r} dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\frac{(2-r)^2}{2}\,dr\,d\theta.$$

3

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \frac{r}{2}(r^2-4r+4)\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^1 (r^3-4r^2+4r)\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 (r^3-4r^2+4r)\,dr$$

$$= \frac{1}{2}(\theta)_0^{\frac{\pi}{2}}\left(\frac{r^4}{4}-\frac{4r^3}{3}+2r^2\right)_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{1}{4}-\frac{1}{3}+2\right) = \frac{\pi}{4}\left(\frac{3-4+24}{12}\right) = \frac{23\pi}{48}.$$

2)

- $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-(f(x))^2}}dx = -\frac{1}{a} sech^{-1}\left(\frac{f(x)}{a}\right) + c,\ (0 < f(x) < a)$

- $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{(f(x))^2+a^2}}dx = -\frac{1}{a} csch^{-1}\left(\frac{f(x)}{a}\right) + c,\ \ (f(x) \neq 0)$

Trigonometric and hyperbolic formulas

- $\sin^2 x + \cos^2 x = 1$, $\sec^2 x - \tan^2 x = 1$, $\csc^2 x - \cot^2 x = 1$
- $\cosh^2 x - \sinh^2 x = 1$, $sech^2 x + tanh^2 x = 1$, $coth^2\ x - csch^2 = 1$
- $\cos^2 x = \dfrac{1+\cos(2x)}{2}$, $\sin^2 x = \dfrac{1-\cos(2x)}{2}$
- $\sin ax \cos bx = \frac{1}{2}[\sin(ax+bx) + \sin(ax-bx)]$
- $\sin ax \sin bx = \frac{1}{2}[\cos(ax-bx) - \cos(ax+bx)]$
- $\cos ax \cos bx = \frac{1}{2}[\cos(ax+bx) + \cos(ax-bx)]$

Usual infinite series

- $\displaystyle\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{n=1}^{\infty}(-1)^n x^n$
- $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=1}^{\infty} x^n$
- $\displaystyle e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

5) $0 \le \rho \le 2$; $0 \le \theta \le 2\pi$; $0 \le \phi \le \frac{\pi}{2}$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx.$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2} \rho^2 \cdot \rho \cos\phi \sin\phi \, d\rho\, d\theta\, d\phi.$$

$$= \int_0^{\frac{\pi}{2}} \underset{f'}{\cos\phi}\, \underset{f}{\sin\phi}\, d\phi \int_0^{2\pi} d\theta \int_0^{2} \rho^3 d\rho.$$

3

$$= \left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}} [\theta]_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^{2}$$

$$= \left(\frac{\sin^2\frac{\pi}{2}}{2} - \frac{\sin^2 0}{2}\right)\left(\frac{\pi}{2}\right)(4)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right).4 = \pi.$$

Q_2) 17

1) $S_n = a\,\frac{1-r^n}{1-r}$. 2

$S_9 = -3\,\frac{1-(\frac{1}{2})^9}{1-\frac{1}{2}} = (-3)(2)(1-(\frac{1}{2})^9) \simeq -5{,}9882..$

2) $a_n = \frac{5n}{e^{2n}}$, let $f(x) = \frac{5x}{e^{2x}}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{5x}{e^{2x}} = \lim_{x\to\infty}\frac{5}{2e^{2x}} = 0.$$

2

$\Rightarrow \lim_{n\to\infty} a_n = 0$

3). $a_n = \frac{3n^3-6}{2n^3-6}$; $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^3-6}{2n^3-6} = \lim_{n\to\infty}\frac{3n^3}{2n^3} = \lim_{n\to\infty}\frac{3}{2} = \frac{3}{2} \neq 0$

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{3n^3-6}{2n^3-6}\right)$ diverges. 2

3)

4) $\sum_{n=1}^{\infty}\left(\frac{2}{3^n}-\frac{3}{4^n}\right)$.

$\sum_{n=1}^{\infty}\frac{2}{3^n}$ converges, geometric sequence, $r=\frac{1}{3}$. $|r|<1$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{2}{3^n}=\frac{2}{1-(\frac{1}{3})}=\frac{2}{\frac{2}{3}}=3.$$

$\sum_{n=1}^{\infty}\frac{3}{4^n}$ converges, geometric sequence, $r=\frac{1}{4}$. $|r|<1$. 3

$$\Rightarrow \sum_{n=1}^{\infty}\frac{3}{4^n}=\frac{3}{1-\frac{1}{4}}=\frac{3}{\frac{3}{4}}=4.$$

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{2}{3^n}-\frac{3}{4^n}\right)$ converges and $\sum_{n=1}^{\infty}\left(\frac{2}{3^n}-\frac{3}{4^n}\right)=3-4=-1$.

5)

Let $f(x)=xe^{-x^2}$ $\forall x\geq 1$.

- f is a positive valued function.
- f is continuous

$f'(x)=e^{-x^2}+x(-2x)e^{-x^2}=e^{-x^2}(1-2x^2)\leq 0$

$\Rightarrow f$ is decreasing.

$$\int_1^{\infty}xe^{-x^2}dx=\lim_{t\to\infty}\int_1^t xe^{-x^2}dx=\lim_{t\to\infty}\left[\frac{1}{2}e^{-x^2}\right]_1^t$$
$$=\lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2}-\frac{1}{2}e^{-1}\right]$$
$$=-\frac{1}{2}e^{-1}.$$

2

$\Rightarrow \int_1^{\infty}xe^{-x^2}dx$ converges

$\xRightarrow{I.T} \sum_{n=1}^{\infty}ne^{-n^2}$ converges.

4)

6) $a_n = \dfrac{n^2+2n}{n^3+1}$, let. $b_n = \dfrac{n^2}{n^3} = \dfrac{1}{n}$.

$\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges. Harmonic.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n}{n^3+1} \cdot n = \lim_{n\to\infty} \frac{n^3}{n^3} = \lim_{n\to\infty} 1 = 1 > 0$$

3

$\xrightarrow{L.C.T} \sum_{n=1}^{\infty} a_n$ diverges.

7)

$$a_n = \frac{2^{n-1}}{5^n(n+1)}.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n}}{5^{n+1}(n+2)} \cdot \frac{5^n(n+1)}{2^{n-1}}\right|$$

$$= \lim_{n\to\infty} \frac{2}{5} \cdot \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{2}{5} \cdot \frac{n}{n} = \lim_{n\to\infty} \frac{2}{5} = \frac{2}{5} < 1.$$

$\xrightarrow{ART.} \sum_{n=1}^{\infty} a_n$ converges.

3

5)

Q3) 9

1) $u_n = \frac{(-3)^n x^n}{n}$.

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{3^{n+1}}^{3}\,\cancel{x^{n+1}}^{x}}{n+1} \cdot \frac{n}{\cancel{3^n}\,\cancel{x^n}}\right|$.

$= \lim_{n\to\infty} \left|\frac{3xn}{n+1}\right| = \lim_{n\to\infty} \frac{3n}{n+1}|x|$

$= \lim_{n\to\infty} 3|x| = 3|x| < 1$.

$\Rightarrow |x| < \frac{1}{3}$. $\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$.

$\Rightarrow$ the radius $r = \frac{1}{3}$.

- $x = \frac{1}{3}$; $u_n = \frac{(-3)^n\left(\frac{1}{3}\right)^n}{n} = \frac{(-1)^n}{n}$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

$a_n = \frac{1}{n}$,

$k \leq k+1 \Rightarrow \frac{1}{k} \geq \frac{1}{k+1} \Rightarrow a_k \geq a_{k+1}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\xrightarrow{\text{A.S.T}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

- $x = -\frac{1}{3}$; $u_n = \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{n} = \frac{1}{n}$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (Harmonic)

$\Rightarrow$ the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$.

b+

6)

2)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots$$

$$\Rightarrow \frac{x}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+1} = x + x^3 + x^5 + \cdots + x^{2n+1} + \cdots$$

$$(\ln(1-x^2))' = \frac{-2x}{1-x^2} = -2 \sum_{n=1}^{\infty} x^{2n+1}$$

$$= -2x - 2x^3 - 2x^5 - \cdots - 2x^{2n+1} + \cdots$$

$$\Rightarrow \ln(1-x^2) = -2 \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} = -\sum_{n=1}^{\infty} \frac{x^{2n+2}}{n+1}.$$

3

$$= -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots - \frac{x^{2n+2}}{n+1} + \cdots$$

3).

$$(\tan^{-1}(x^2))' = \frac{2x}{1+x^4}.$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n} = 1 - x^4 + x^8 - x^{12} + \cdots + (-1)^n x^{4n} + \cdots$$

$$\frac{2x}{1+x^4} = 2\sum_{n=0}^{\infty} (-1)^n x^{4n+1} = 2x - 2x^5 + 2x^9 - 2x^{13} + \cdots + 2(-1)^n x^{4n+1} + \cdots$$

$$\tan^{-1}(x^2) = \frac{2x^2}{2} - \frac{2x^6}{6} + 2\frac{x^{10}}{10} - 2\frac{x^{14}}{14} + \cdots + 2(-1)^n \frac{x^{4n+2}}{4n+2} + \cdots$$

3

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots + (-1)^n \frac{x^{4n+2}}{2n+1} + \cdots$$

7)