Department of Mathematics

Final Examination	Math 204	Semester I - 1446	Time: 3H

Question 1 : [4,4,5]

a) Solve the initial value problem $\begin{cases} xdy = \left(y + \frac{4x^5}{1+x^4}\right) dx \\ y(2) = 3. \end{cases}$

b) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{ye^x + 3e^x - y - 3}{ye^x - 2e^x + 4y - 8}$.

c) Find the orthogonal trajectories for the family of curves represented by $y = e^{Cx}$.

Question 2 : [4,4]

a) Obtain the general solution of the differential equation

$$(9 - 6y + e^{-3x})dx = 2dy.$$

b) Find the general solution of the differential equation

$$y'' - 4y = xe^x + \cos 2x.$$

Question 3 : [4,5]

a) Solve the system of differential equations $\begin{cases} x'' - y = t^3 \\ y'' - x = 0. \end{cases}$

b) Find the general series solution of the differential equation about the ordinary point $x_0 = 0$, such that $y(0) = 0$ and $y'(0) = 0$

$$(x^2 - 1)y'' + 6xy' + 4y = -4.$$

Question 4 : [5,5]

a) Find the Fourier series for the 2π-periodic function

$$f(x) = \begin{cases} -(\pi + x) & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } \quad 0 \le x < \pi \end{cases}$$

Then deduce the value of the numerical series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

b) Find the Fourier integral representation for the function

$$f(x) = \begin{cases} \pi - |x| \,, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$$

and deduce the value of the integral $\displaystyle\int_0^\infty \left(\frac{\sin \pi x}{x}\right)^2 dx.$

Solution of the final examination — Math 204 Semester I — December 2024

Question 1 : [4,4,5]

a) We take $x > 0$. $y' - \frac{1}{x}y = \frac{4x^4}{1+x^4}$, $P(x) = -\frac{1}{x}$, $Q(x) = \frac{4x^4}{1+x^4}$.

$\int P(x)dx = \ln(\frac{1}{x})$, then $\mu(x) = \frac{1}{x}$

$y(x) = \frac{1}{\mu(x)} \int \mu(t)\, Q(t)\, dt$

$$y = x\ln(1+x^4) + cx.$$

$y(2) = 3$, then $c = \frac{1}{2}(3 - 2\ln(17))$.

b) We have $y' = \dfrac{(y+3)(e^x-1)}{(y-2)(e^x+4)}$, then $\frac{y-2}{y+3}y' = \frac{e^x-1}{e^x+4}$ and

$\int \frac{y-2}{y+3}\,dy = \int \frac{e^x-1}{e^x+4}\,dx = \int\left[1 - \frac{5}{e^x+4}\right]dx = x - 5\int \frac{e^{-x}}{1+4e^{-x}}dx$

$\int\left[1 - \frac{5}{y+3}\right]dx =$

$$y - 5\ln|y+3| - \frac{5}{4}\ln(4e^{-x}+1) - x = c.$$

c) $y = e^{Cx}$, then $y' = Cy = \frac{y\ln y}{x}$. The equation of the orthogonal trajectories is $y'y\ln y = -x$. Hence the solution of the equation is

$y' = \frac{y\ln y}{x}$

so DE of trajec with is $y = \frac{-x}{y\ln y}$

$\int y \ln y\, dy = \int -x\, dx$

$$2y^2\ln y - y^2 = -2x^2 + c.$$

Question 2 : [4,4]

a) $y' + 3y = \frac{1}{2}(9 + e^{-3x})$. The general solution is $y = (a+bx)e^{-3x} + c$.

linear $\mu(x) = e^{3x}$

$$y = (a + \frac{x}{2})e^{-3x} + \frac{3}{2}.$$

b) The characteristic equation of the differential equation is $r^2 - 4$. The general solution of the differential equation is

$$y = ae^{2x} + be^{-2x} + (cx+d)e^x + A\cos(2x) + B\sin(2x).$$

Hence, $y = ae^{2x} + be^{-2x} - (\frac{1}{3}x + \frac{2}{9})e^x - \frac{1}{8}\cos(2x)$.

Question 3 : [4,5]

a) The operator form of the system $\begin{cases} D^2x - y &= t^3 \\ -x + D^2y &= 0 \end{cases}$

We apply D^2, we get $x^{(4)} - x = t$, then $x = ae^t + be^{-t} + c\cos t + d\sin t + At + B$.

$$x(t) = ae^t + be^{-t} + c\cos t + d\sin t - 6t$$

and

$$y(t) = ae^t + be^{-t} - c\cos t - d\sin t - t^3.$$

b) Let $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, $xy'(x) = \sum_{n=0}^{+\infty} n a_n x^n$, $x^2 y'' = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$,

$y'' = \sum_{n=0}^{+\infty} (n+1)(n+2)a_{n+2}x^n$.

Hence

$$\sum_{n=0}^{+\infty} \left(n(n-1)a_n - (n+1)(n+2)a_{n+2} + 6na_n + 4a_n\right) x^n = -4.$$

For $n = 0$, we get $a_2 = 2(a_0 + 1)$, and for $n \geq 1$,

$$a_{n+2} = \frac{n+4}{n+2} a_n.$$

If $U_n = a_{2n}$ and $V_n = a_{2n-1}$, for $n \geq 1$, we have

$$U_n = a_{2n} = \frac{n+1}{n} U_{n-1}, \Rightarrow U_n = a_{2n} = (n+1)(a_0 + 1)$$

and

$$V_n = a_{2n-1} = \frac{2n+3}{2n+1} V_{n-1}, \Rightarrow V_n = a_{2n-1} = \frac{1}{3}(2n+3)a_1.$$

As $a_0 = 0$ and $a_1 = 0$,

$$y = \sum_{n=1}^{+\infty} (n+1)x^{2n} = \frac{x^2(2 - x^2)}{(1 - x^2)^2}.$$

Question 4 : [5,5]

a) The function f is odd, the $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)dx = \frac{2}{n}$.
Hence the Fourier series of f is

$$\sum_{n=1}^{+\infty} \frac{2}{n}\sin(nx).$$

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As f is differentiable on $(0, \pi)$, we get for $x = \frac{\pi}{2}$,

$$\frac{\pi}{2} = \sum_{n=1}^{+\infty} \frac{2}{n}\sin(n\frac{\pi}{2}) = \sum_{n=0}^{+\infty} \frac{2(-1)^n}{2n+1}.$$

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So $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\right) = \frac{\pi}{4}$

b) The function f is even, then the Fourier integral of f is

$$\int_0^{+\infty} A(t)\cos(xt)dt$$

where $A(t) = \frac{2}{\pi}\int_0^{+\infty}(\pi - x)\cos(xt)dx = \frac{2}{\pi}\frac{1-\cos(\pi t)}{t^2}$.
Taking $x + 0$, we get

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$$\frac{\pi^2}{4} = \int_0^{+\infty} \frac{1}{t^2}\sin^2(\frac{\pi t}{2})dt.$$

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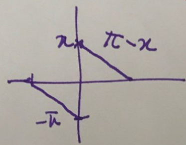
$$= \int_0^{+\infty} 2\,\frac{\sin^2(x\pi)}{(2x)^2}dx$$

On substituting $x = \frac{t}{2}$, we get the desired result.

$$\frac{\pi^2}{4} = \frac{1}{2}\int_0^{+\infty}\left(\frac{\sin(\pi x)}{x}\right)^2 dx$$

So $\boxed{\int_0^{+\infty}\left(\frac{\sin \pi x}{x}\right)^2 dx = \frac{\pi^2}{2}}$

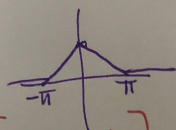
(1)



$$f(x) = \frac{1}{\pi}\int_0^{+\infty} A(\lambda)\cos(\lambda x)\,d\lambda$$

$$A(\lambda) = 2\int_0^{\pi}(\pi - x)\cos(\lambda x)\,dx$$

(2)



$u(x) = \pi - x \Rightarrow u'(x) = -1$

$v'(x) = \cos(\lambda x)$, $v(x) = \frac{\sin(\lambda x)}{\lambda}$

$$A(\lambda) = 2\left[\left[\frac{(\pi - x)\sin(\lambda x)}{\lambda}\right]_0^{\pi} + \frac{1}{\lambda}\int_0^{\pi}\sin(\lambda x)dx\right]$$

$$= \frac{2}{\lambda}\int_0^{\pi}\sin(\lambda x)dx = \frac{2}{\lambda^2}\,-\left[\cos\lambda x\right]_0^{\pi}$$

$$= \frac{-2}{\lambda^2}\left[\cos\lambda\pi - 1\right]$$