

Time: 3 hours

Instructions: Calculators are not allowed.

Question 1 (3+2+2 Marks):

- Find the value of c that satisfies the mean value theorem for definite integrals for the function $f(x) = \frac{1}{\sqrt{x+1}}$ on the interval $[3, 8]$.
- If $F(x) = \int_{\sinh x}^{2^{x+1}} (1+t^2) dt$, find $F'(x)$.
- If $y(x) = (\cos x)^x + e^{-5x}$, find $\frac{dy}{dx}$.

Question 2: Evaluate the following integrals (2+3+2+3+3+2 Marks)

- $\int \frac{1}{\sqrt{x^2-4x}} dx$
- $\int x^2 \cos(3x) dx$
- $\int \frac{(\sqrt{x}-1) \sec(\sqrt{x}-\ln \sqrt{x})}{x} dx$
- $\int \frac{5x^2+6x+4}{(x+1)(x^2+x+1)} dx$
- $\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx$
- $\int \tan^3 x \sec^2 x dx$.

Question 3 (2+2+3+3+2+2+4 Marks):

- Calculate $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.
- Determine whether the improper integral $\int_0^{\infty} \frac{x}{(1+x^2)^2} dx$ converges or diverges.
- Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find its area.
- Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = x+2$, then find the volume of the solid generated by revolving this region about the x -axis.
- Find the arc length of $y = \frac{2}{3}x\sqrt{x}$, from $x = 0$ to $x = 1$.
- Convert the polar equation $r = \cos \theta + \sec \theta$ into a Cartesian equation.
- Sketch the common region between the curves $r = 2$ and $r = 2 + 2 \cos \theta$, then find its area.

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Final Exam
Dr Tariq A. Alfadhel

Question (1): [3 + 2 + 2 = 7 marks]

- Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = \frac{1}{\sqrt{x+1}}$ on the interval $[3, 8]$.

Solution: Using the formula $(b-a) f(c) = \int_a^b f(x) dx$.

$$(8-3) \frac{1}{\sqrt{c+1}} = \int_3^8 \frac{1}{\sqrt{x+1}} dx = \int_3^8 (x+1)^{-\frac{1}{2}} dx$$

$$\frac{5}{\sqrt{c+1}} = \left[\frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}} \right]_3^8 = 2\sqrt{8+1} - 2\sqrt{3+1} = 2(3) - 2(2) = 6 - 4 = 2$$

$$\frac{5}{\sqrt{c+1}} = 2 \implies \sqrt{c+1} = \frac{5}{2} \implies c+1 = \frac{25}{4} \implies c = \frac{25}{4} - 1 = \frac{21}{4}$$

The desired value is $c = \frac{21}{4} \in (3, 8)$.

- Find $F'(x)$, if $F(x) = \int_{\sinh x}^{2^{x+1}} (1+t^2) dt$.

Solution:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_{\sinh x}^{2^{x+1}} (1+t^2) dt \\ &= \left[1 + (2^{x+1})^2 \right] (2^{x+1} \ln 2) - \left[1 + (\sinh x)^2 \right] (\cosh x) \\ &= 2^{x+1} \ln 2 (1 + 2^{2x+2}) - \cosh x (1 + \sinh^2 x) \\ &= 2^{x+1} \ln 2 (1 + 2^{2x+2}) - \cosh^3 x \end{aligned}$$

- Find y' if $y(x) = (\cos x)^x + e^{-5x}$.

Solution:

$$\begin{aligned} y(x) &= (\cos x)^x + e^{-5x} = e^{\ln|\cos x|^x} + e^{-5x} = e^{x \ln|\cos x|} + e^{-5x} \\ y'(x) &= e^{x \ln|\cos x|} \left[(1) \ln|\cos x| + x \left(\frac{-\sin x}{\cos x} \right) \right] + e^{-5x} (-5) \\ &= (\cos x)^x [\ln|\cos x| - x \tan x] - 5e^{-5x} \end{aligned}$$

Question (2): [2 + 3 + 2 + 3 + 3 + 2 = 15 marks]

Evaluate the following integrals :

1. $\int \frac{1}{\sqrt{x^2 - 4x}} dx$.

Solution: By completing the square.

$$x^2 - 4x = (x^2 - 4x + 4) - 4 = (x - 2)^2 - (2)^2$$

$$\int \frac{1}{\sqrt{x^2 - 4x}} dx = \int \frac{1}{\sqrt{(x - 2)^2 - (2)^2}} dx = \cosh^{-1} \left(\frac{x - 2}{2} \right) + c .$$

2. $\int x^2 \cos(3x) dx$.

Solution: Using integration by parts .

$$\begin{aligned} u &= x^2 & dv &= \cos(3x) dx \\ du &= 2x dx & v &= \frac{1}{3} \sin(3x) \end{aligned}$$

$$\begin{aligned} \int x^2 \cos(3x) dx &= x^2 \left(\frac{1}{3} \sin(3x) \right) - \int \frac{1}{3} \sin(3x) 2x dx \\ &= \frac{x^2}{3} \sin(3x) - \frac{2}{3} \int x \sin(3x) dx \end{aligned}$$

Using integration by parts again .

$$\begin{aligned} u &= x & dv &= \sin(3x) dx \\ du &= dx & v &= -\frac{1}{3} \cos(3x) \end{aligned}$$

$$\begin{aligned} \int x^2 \cos(3x) dx &= \frac{x^2}{3} \sin(3x) - \frac{2}{3} \left[x \left(-\frac{1}{3} \cos(3x) \right) - \int -\frac{1}{3} \cos(3x) dx \right] \\ &= \frac{x^2}{3} \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{2}{9} \int \cos(3x) dx \\ &= \frac{x^2}{3} \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{2}{27} \sin(3x) + c \end{aligned}$$

3. $\int \frac{(\sqrt{x} - 1) \sec(\sqrt{x} - \ln \sqrt{x})}{x} dx$

Solution:

$$\begin{aligned} \int \frac{(\sqrt{x} - 1) \sec(\sqrt{x} - \ln \sqrt{x})}{x} dx &= \int \sec \left(\sqrt{x} - \ln \left| x^{\frac{1}{2}} \right| \right) \left(\frac{\sqrt{x} - 1}{x} \right) dx \\ &= \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{\sqrt{x}}{x} - \frac{1}{x} \right) dx = \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right) dx \\ &= 2 \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left[\frac{1}{2} \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right) \right] dx \end{aligned}$$

$$\begin{aligned}
&= 2 \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x} \right) dx \\
&= 2 \ln \left| \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) + \tan \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \right| + c .
\end{aligned}$$

$$4. \int \frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} dx .$$

Solution : Using the method of partial fractions.

$$\begin{aligned}
\frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} &= \frac{A}{x+1} + \frac{Bx + C}{x^2 + x + 1} \\
\frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} &= \frac{A(x^2 + x + 1)}{(x+1)(x^2 + x + 1)} + \frac{(Bx + C)(x+1)}{(x^2 + x + 1)(x+1)} \\
5x^2 + 6x + 4 &= Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \\
5x^2 + 6x + 4 &= (A+B)x^2 + (A+B+C)x + (A+C)
\end{aligned}$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{aligned}
A + B &= 5 & \longrightarrow (1) \\
A + B + C &= 6 & \longrightarrow (2) \\
A + C &= 4 & \longrightarrow (3)
\end{aligned}$$

Equation (2) - Equation (1) : $C = 1$.

From Equation (3) : $A + 1 = 4 \implies A = 3$.

From Equation (1) : $3 + B = 5 \implies B = 2$.

$$\begin{aligned}
\int \frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} dx &= \int \left(\frac{3}{x+1} + \frac{2x+1}{x^2 + x + 1} \right) dx \\
&= 3 \int \frac{1}{x+1} dx + \int \frac{2x+1}{x^2 + x + 1} dx = 3 \ln |x+1| + \ln |x^2 + x + 1| + c
\end{aligned}$$

$$5. \int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx .$$

Solution : Using trigonometric substitutions.

$$\text{Put } x = 2 \sin \theta \implies \sin \theta = \frac{x}{2} .$$

$$dx = 2 \cos \theta d\theta .$$

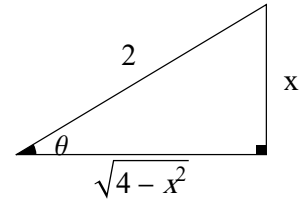
$$\begin{aligned}
(4-x^2)^{\frac{3}{2}} &= (4-4\sin^2 \theta)^{\frac{3}{2}} = [4(1-\sin^2 \theta)]^{\frac{3}{2}} = [2^2 \cos^2 \theta]^{\frac{3}{2}} \\
&= (2^2)^{\frac{3}{2}} (\cos^2 \theta)^{\frac{3}{2}} = 2^3 \cos^3 \theta \\
\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx &= \int \frac{4 \sin^2 \theta \cdot 2 \cos \theta}{2^3 \cos^3 \theta} d\theta = \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta
\end{aligned}$$

$$= \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + c$$

$$\sin \theta = \frac{x}{2} \implies \theta = \sin^{-1} \left(\frac{x}{2} \right).$$

From the triangle :

$$\tan \theta = \frac{x}{\sqrt{4-x^2}}$$



$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1} \left(\frac{x}{2} \right) + c.$$

6. $\int \tan^3 x \sec^2 x dx$.

Solution : .

$$\int \tan^3 x \sec^2 x dx = \int (\tan x)^3 \sec^2 x dx = \frac{(\tan x)^4}{4} + c = \frac{\tan^4 x}{4} + c .$$

Question (3): [2 + 2 + 3 + 3 + 2 + 2 + 4 = 18 marks]

1. Calculate $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

Solution:

$$\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} e^{\ln|x|^{\frac{1}{x}}} = \lim_{x \rightarrow \infty} e^{\frac{1}{x} \ln|x|} = \lim_{x \rightarrow \infty} e^{\frac{\ln|x|}{x}}$$

$$\lim_{x \rightarrow \infty} \frac{\ln|x|}{x} \quad \left(\frac{\infty}{\infty} \right)$$

Using L'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0 .$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\ln|x|}{x} = 0 .$$

$$\text{So, } \lim_{x \rightarrow \infty} x^{\frac{1}{x}} = e^0 = 1 .$$

2. Determine whether the improper integral $\int_0^{\infty} \frac{x}{(1+x^2)^2} dx$ converges or diverges.

Solution:

$$\int_0^{\infty} \frac{x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \int_0^t (1+x^2)^{-2} (2x) dx \right)$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{(1+x^2)^{-1}}{-1} \right]_0^t \right) = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{-1}{1+x^2} \right]_0^t \right) \\
&= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{-1}{1+t^2} - \frac{-1}{1+(0)^2} \right] \right) = \frac{1}{2} [0 - (-1)] = \frac{1}{2} .
\end{aligned}$$

Hence, the improper integral converges.

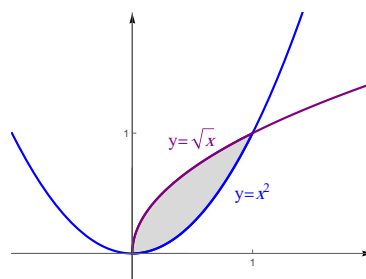
3. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find its area.

Solution:

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = \sqrt{x}$ represents the upper half of a parabola opens to the right and its vertex is $(0, 0)$.

Points of intersection of $y = x^2$ and $y = \sqrt{x}$:
 $x^2 = \sqrt{x} \implies x^4 = x$
 $\implies x^4 - x = 0$
 $\implies x(x^3 - 1) = 0$
 $\implies x = 0, x = 1$.



$$\begin{aligned}
A &= \int_0^1 (\sqrt{x} - x^2) dx = \left[2\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^1 \\
&= \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \frac{1}{3} .
\end{aligned}$$

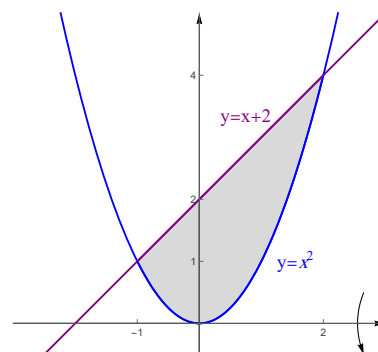
4. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = x + 2$, then find the volume of the solid generated by revolving this region about the x -axis.

Solution :

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = x + 2$ represents a straight line passing through $(0, 2)$ with slope 1.

Points of intersection of $y = x^2$ and $y = x + 2$:
 $x^2 = x + 2 \implies x^2 - x - 2 = 0$
 $\implies (x + 1)(x - 2) = 0$
 $\implies x = -1, x = 2$



Using Washer Method :

$$\begin{aligned}
 V &= \pi \int_{-1}^2 \left[(x+2)^2 - (x^2)^2 \right] dx \\
 &= \pi \int_{-1}^2 [x^2 + 4x + 4 - x^4] dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} + 2x^2 + 4x \right]_{-1}^2 \\
 &= \pi \left[\left(-\frac{2^5}{5} + \frac{2^3}{3} + 2(2^2) + 4(2) \right) - \left(-\frac{(-1)^5}{5} + \frac{(-1)^3}{3} + 2((-1)^2) + 4(-1) \right) \right] \\
 &= \pi \left[\left(-\frac{32}{5} + \frac{8}{3} + 8 + 8 \right) - \left(\frac{1}{5} - \frac{1}{3} + 2 - 4 \right) \right] \\
 &= \pi \left(-\frac{33}{5} + \frac{9}{3} + 18 \right) = \pi \left(-\frac{33}{5} + 21 \right) = \frac{72\pi}{5} .
 \end{aligned}$$

5. Find the arc length of $y = \frac{2}{3}x\sqrt{x}$, from $x = 0$ to $x = 1$.

Solution :

$$y = \frac{2}{3}x\sqrt{x} = \frac{2}{3} x^{\frac{3}{2}} .$$

$$y' = \frac{2}{3} \left(\frac{3}{2} x^{\frac{1}{2}} \right) = x^{\frac{1}{2}} .$$

$$\begin{aligned}
 L &= \int_0^1 \sqrt{1 + \left(x^{\frac{1}{2}} \right)^2} dx = \int_0^1 \sqrt{1 + x} dx = \int_0^1 (1 + x)^{\frac{1}{2}} dx \\
 &= \left[\frac{2}{3} (1 + x)^{\frac{3}{2}} \right]_0^1 = \frac{2}{3} (1 + 1)^{\frac{3}{2}} - \frac{2}{3} (1 + 0)^{\frac{3}{2}} = \frac{2}{3} (\sqrt{8} - 1) .
 \end{aligned}$$

6. Convert the polar equation $r = \cos \theta + \sec \theta$ into a Cartesian equation.

Solution:

$$r = \cos \theta + \sec \theta \implies r^2 = r \cos \theta + r \sec \theta = r \cos \theta + \frac{r}{\cos \theta}$$

$$\implies x^2 + y^2 = x + \frac{r}{\frac{x}{r}} = x + \frac{r^2}{x} = x + \frac{x^2 + y^2}{x}$$

$$\implies x^3 + xy^2 = x^2 + x^2 + y^2 \implies x^3 + xy^2 - 2x^2 - y^2 = 0$$

7. Sketch the common region between the curves $r = 2$ and $r = 2 + 2 \cos \theta$, then find its area.

Solution:

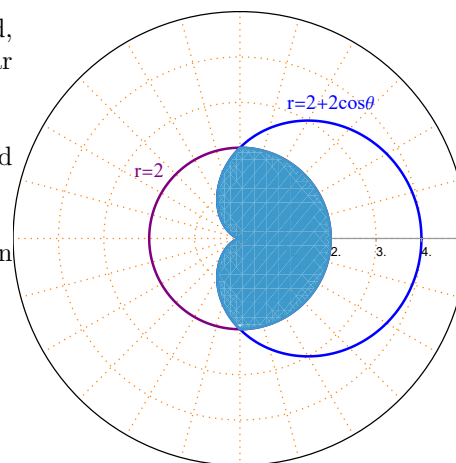
$r = 2 + 2 \cos \theta$ represents a cardioid, symmetric with respect to the polar axis.

$r = 2$ represents a circle centered at the pole, and its radius is 2.

Points of intersection between $r = 2 + 2 \cos \theta$ and $r = 2$:

$$2 + 2 \cos \theta = 2 \implies 2 \cos \theta = 0$$

$$\implies \cos \theta = 0 \implies \theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$



Note that the shaded region is symmetric with respect to the polar axis.

$$\begin{aligned} A &= 2 \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} (2)^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (2 + 2 \cos \theta)^2 d\theta \right) \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8 \cos \theta + 4 \left(\frac{1 + \cos 2\theta}{2} \right) \right) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8 \cos \theta + (2 + 2 \cos 2\theta)) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (6 + 8 \cos \theta + 2 \cos 2\theta) d\theta \\ &= [4\theta]_0^{\frac{\pi}{2}} + [6\theta + 8 \sin \theta + \sin 2\theta]_{\frac{\pi}{2}}^{\pi} \\ &= \left[4 \left(\frac{\pi}{2} \right) - 4(0) \right] + \left[(6\pi + 8 \sin(\pi) + \sin(2\pi)) - \left(6 \left(\frac{\pi}{2} \right) + 8 \sin \left(\frac{\pi}{2} \right) + \sin(\pi) \right) \right] \\ &= [2\pi - 0 + [(6\pi + 0 + 0) - (3\pi + 8 + 0)]] = 2\pi + (3\pi - 8) = 5\pi - 8. \end{aligned}$$