King Saud University
Faculty of Sciences
Department of Mathematics

جامعة الملك سعود
King Saud University
كلية العلوم

Final Exam	Math 106	December 2024

Question 1 : (2+3+2)

1. Find $\displaystyle\int \frac{\left(1+x^{\frac{2}{3}}\right)^4}{x^{\frac{1}{3}}}dx.$

2. Use Riemann sums to find $\displaystyle\int_0^2 (x^2+4)dx.$

3. Compute $\displaystyle\int \frac{\ln x+1}{\sqrt{9-x^2(\ln x)^2}}dx.$

Question 2 : (3+3+3)

1. Find $\displaystyle\int \frac{dx}{x\sqrt{1-x^8}}.$

2. Compute $\displaystyle\lim_{x\to+\infty}(1+e^{-2x})^{e^x}.$

3. Evaluate $\displaystyle\int x\sec^2 x dx.$

Question 3 : (3+3+3)

1. Compute the integral $\displaystyle\int \sin^2 x\cos^5 x dx.$

2. Find $\displaystyle\int \frac{x+3}{(x^2+9)^{\frac{3}{2}}}dx.$

3. Evaluate $\int \frac{x^3+3}{(x+1)(x^2+1)}dx$.

Question 4 : (3+3+3)

1. Find $\int_3^{+\infty} \frac{dx}{x(\ln x)^3}$.

2. Sketch the region bounded by the curves $y = \sqrt{x+6}$, the $x-$axis, $y = x$, and find its area.

3. Find the volume of the solid obtained by revolving the region bounded by $y = (x-2)^2$, $y = 1$ about the $y-$axis.

Question 5 : (3+3)

1. Find the surface area obtained by revolving the curve $x = t^3$, $y = 3t+1, 0 \le t \le 1$ about the $y-$axis.

2. Sketch the region inside $r = 3\sin\theta$ and outside $r = 3 - 3\sin\theta$ and find its area.

Solution of the Final Exam	Math 106	December 2024

Question 1 :

1.

$$\int \frac{(1+x^{\frac{2}{3}})^4}{x^{\frac{1}{3}}}dx \overset{u=1+x^{\frac{2}{3}}}{=} \frac{3}{2}\int u^4 du = \frac{3}{10}(1+x^{\frac{2}{3}})^5 + c \quad (\mathbf{1})+(\mathbf{1})$$

2.

$$\begin{aligned}\int_0^2 (x^2+4)dx &= \lim_{n\to+\infty} \frac{2}{n}\sum_{k=1}^{n}(\frac{4k^2}{n^2}+4)\\ &= \lim_{n\to+\infty} \frac{2}{n}\left(\frac{1}{n^2}(\frac{4n(n+1)(2n+1)}{6})+4n\right) \quad (\mathbf{2})\\ &= \lim_{n\to+\infty} \frac{4}{3}(1+\frac{1}{n})(2+\frac{1}{n})+8 = \frac{32}{3}. \quad (\mathbf{1})\end{aligned}$$

3. $\displaystyle\int \frac{\ln x+1}{\sqrt{9-x^2(\ln x)^2}}dx \overset{u=x\ln x}{=} \int \frac{du}{\sqrt{9-u^2}}dx = \sin^{-1}(\frac{x\ln x}{3})+c \quad (\mathbf{1})+(\mathbf{1}).$

Question 2 :

1. $\displaystyle\int \frac{dx}{x\sqrt{1-x^8}} \overset{u=x^4}{=} \frac{1}{4}\int \frac{du}{u\sqrt{1-u^2}} = -\frac{1}{4}\text{sech}^{-1}(x^4)+c \quad (\mathbf{2})+(\mathbf{1}).$

2. $\displaystyle\lim_{x\to+\infty}(1+e^{-2x})^{e^x} = \lim_{x\to+\infty} e^{\frac{\ln(1+e^{-2x})}{e^{-x}}} = 1 \quad (\mathbf{1})+(\mathbf{2}).$

3. By parts $\displaystyle\int x\sec^2 x dx = x\tan x + \ln|\cos x| + c \quad (\mathbf{1.5})+(\mathbf{1.5}).$

Question 3 :

1.

$$\begin{aligned}\int \sin^2 x\cos^5 x dx &\overset{u=\sin x}{=} \int u^2(1-u^2)^2 du = \int u^6+u^2-2u^4 du \quad (\mathbf{2})\\ &= \frac{1}{7}\sin^7 x - \frac{2}{5}\sin^5 x + \frac{1}{3}\sin^3 x + c \quad (\mathbf{1})\end{aligned}$$

2.

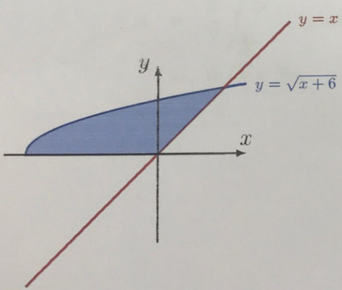
$$\int \frac{x+3}{(x^2+9)^{3/2}}dx \overset{x=3\tan\theta}{=} \frac{1}{3}\int(\sin\theta+\cos\theta)\,d\theta$$
$$= \frac{1}{3}(\sin\theta-\cos\theta)+c$$
$$= \frac{1}{3}\left(\frac{x}{\sqrt{x^2+9}}-\frac{3}{\sqrt{x^2+9}}\right)+c \quad (1.5)$$

3.
$$\frac{x^3+3}{(x+1)(x^2+1)} = 1+\frac{2-x^2-x}{(x+1)(x^2+1)}$$
$$= 1+\frac{1}{x+1}+\frac{-2x+1}{x^2+1} \quad (1.5)$$

$$\int \frac{x^3+3}{(x+1)(x^2+1)}dx = x+\ln|x+1|-\ln(1+x^2)+\tan^{-1}(x)+c. \quad \mathbf{(1.5)}$$

Question 4 :

1. $$\int_3^{+\infty}\frac{dx}{x(\ln x)^3} \overset{u=\ln x}{=} \int_{\ln 3}^{+\infty}\frac{du}{u^3} = \frac{1}{2(\ln 3)^2}. \quad \mathbf{(1.5)+(1.5)}$$



$$A=\int_0^3[y-(y^2-6)]\,dy$$

$$A=\frac{27}{2}$$

2. Graph: **(1)**

$$A=\int_{-6}^0\sqrt{x+6}\,dx+\int_0^3(\sqrt{x+6}-x)\,dx=\frac{27}{2}. \quad \mathbf{(1)+(1)}$$

3.
$$V = 2\pi\int_1^3 x(1-(x-2)^2) \quad \mathbf{(1)}$$
$$\overset{t=x-2}{=} 2\pi\int_{-1}^1(t+2)(1-t^2)\,dt$$

$$V=\pi\int_0^1\left[(\sqrt{y}+2)^2-(2-\sqrt{y})^2\right]dy$$

Scanned with CamScanner

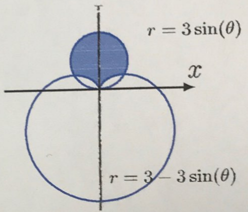
$$= \quad 4\pi \int_{-1}^{1} (1 - t^2)dt \quad (1)$$

$$= \quad \frac{16}{3}\pi \quad (1)$$

Question 5 : (3+3)

1. $A = 2\pi \int_0^1 t^3\sqrt{9t^4+9}dt = \frac{3}{2}\pi \int_0^1 4t^3\sqrt{t^4+1}dt = \pi(2^{\frac{3}{2}}-1).$ **(1.5) + (1.5)**

2. $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6}$ are the solutions of the equation $3 - 3\sin(\theta) = 3\sin(\theta)$. The area inside $r = 3\sin(\theta)$, outside $r = 3 - 3\sin(\theta)$ is:



Graph: (1)

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((3\sin(\theta))^2 - (3 - 3\sin(\theta))^2\right) d\theta \quad (1)$$

$$= 9\sqrt{3} - 3\pi \quad (1)$$