Q

A body is heated to $150°C$ and placed in air at $15°C$. After 1 hour, the temperature of the body becomes $60°C$, find how much additional time is required for the body to cool to $30°C$.

Solution

By using Newton's law of cooling,

$$\frac{dT}{dt} = -k(T - T_s),$$

where T_s is the surrounding temperature.

The solution is

$$T(t) = T_s + (T_0 - T_s)e^{-kt}.$$

Given

$$T_s = 15°C, \qquad T_0 = 150°C,$$

thus

$$T(t) = 15 + 135e^{-kt}.$$

Using the condition that after 1 hour the temperature is $60°C$,

$$60 = 15 + 135e^{-k}.$$

Hence,

$$45 = 135e^{-k},$$

which gives

$$e^{-k} = \frac{1}{3}.$$

Therefore,

$$k = \ln 3.$$

Now find the time when the temperature becomes $30°C$:

$$30 = 15 + 135e^{-kt}.$$

Thus,

$$15 = 135e^{-kt},$$

so

$$e^{-kt} = \frac{1}{9}.$$

Since

$$\frac{1}{9} = 3^{-2},$$

and $k = \ln 3$,

$$e^{-t \ln 3} = 3^{-2}.$$

Therefore,

$$t = 2 \text{ hours}.$$

The body reaches $30°C$ after 2 hours from the beginning.
Since 1 hour has already elapsed, the additional time required is

$$\boxed{1 \text{ hour}}.$$

Q

Initially, there were 100 milligrams (mg) of a radioactive substance. After 6 hours, the mass decreased by 3%. Assuming the rate of decay is proportional to the amount remaining, determine the half-life of the substance.

Solution

we have

$$\frac{dM}{dt} = -kM$$

$$\frac{1}{M}\, dM = -k\, dt$$

$$\ln|M| = -kt + C \Rightarrow M(t) = Ce^{-kt}$$

Use the initial condition $M(0) = 100$ to determine C:

$$M(t) = 100e^{-kt}$$

Using the data $M(6) = 97$

$$97 = 100e^{-6k} \Rightarrow e^{-6k} = \frac{97}{100} = 0.97 \Rightarrow -6k = \ln(0.97) \Rightarrow k = -\frac{\ln(0.97)}{6}$$

$$k \approx \frac{0.03046}{6} \approx 0.005076$$

Compute the half-life

The half-life $T_{1/2}$ is the time at which $M(T_{1/2}) = \frac{100}{2} = 50$. Using the model:

$$50 = 100e^{-kT_{1/2}} \Rightarrow e^{-kT_{1/2}} = 0.5 \Rightarrow -kT_{1/2} = \ln(0.5) \Rightarrow T_{1/2} = \frac{\ln(2)}{k}$$

$$T_{1/2} \approx \frac{0.6931}{0.005076} \approx \boxed{136.5 \text{ hours}}$$

Q

The population of a town grows at a rate proportional to its size. The initial population is 1000, and it increases by 10% in 5 years. What will be the population after 50 years?

Solution

$$\frac{dp}{dt} = kp$$

$$\frac{1}{p}\,dp = k\,dt$$

Integrate both sides:

$$\ln|p| = kt + C$$

Then we have

$$p(t) = Ce^{kt}$$

Apply initial conditions

Given:

$$p(0) = 1000 \Rightarrow C = 1000$$

$$p(5) = 1100$$

So

$$p(t) = 1000e^{kt}$$

Use $p(5) = 1100$ to find k:

$$1100 = 1000e^{5k} \Rightarrow e^{5k} = 1.1 \Rightarrow 5k = \ln(1.1) \Rightarrow k = \frac{\ln(1.1)}{5}$$

Using

$$k \approx \frac{0.09531}{5} = 0.01906$$

Find the population after 50 years

$$p(50) = 1000e^{50k} = 1000e^{50\cdot 0.01906} = 1000e^{0.9531}$$

$$p(50) \approx 1000 \cdot 2.5937 = 2593.7$$

Q

Find the orthogonal trajectories of the family of curves

$$y = x + ce^{-x}$$

Solution

Step 1: Differentiate Eq implicitly with respect to x

we obtain

$$\frac{dy}{dx} = 1 - ce^{-x}$$

Step 2: Eliminate the parameter c From the original equation

we have

$$c = e^{x}(y - x)$$

Substitute into the derivative:

$$\frac{dy}{dx} = 1 - \left(e^{x}(y - x)\right) e^{-x} = 1 - (y - x)$$

Simplify:

$$\frac{dy}{dx} = x - y + 1$$

Step 3: Replacing $\frac{dy}{dx}$ by $-1/\frac{dy}{dx}$

we have

$$\frac{dy}{dx} = \frac{-1}{x - y + 1}$$

Step 4: Solving diff. equation

$$\frac{dy}{dx} = \frac{-1}{x - y + 1}$$

Let $u = x - y + 1$, so $du = dx - dy$, which implies $dy = dx - du$. Substituting:

$$\frac{dx - du}{dx} = \frac{-1}{u} \Rightarrow 1 - \frac{du}{dx} = \frac{-1}{u} \Rightarrow \frac{du}{dx} = 1 + \frac{1}{u}$$

Separate variables:

$$\frac{du}{1+\frac{1}{u}} = dx \Rightarrow \frac{u\,du}{u+1} = dx$$

Let $v = u + 1 \Rightarrow u = v - 1$, $dv = du$, so:

$$\int \frac{(v-1)}{v}\,dv = \int dx \Rightarrow \int \left(1 - \frac{1}{v}\right) dv = x + C \Rightarrow v - \ln|v| = x + C$$

Return to original variables:

$$v = x - y + 2 \Rightarrow x - y + 2 - \ln|x - y + 2| = x + C$$

$$x - y + 2 - \ln|x - y + 2| = x + C$$

Thus, the orthogonal trajectories are given implicitly by:

$$y = 2 + C1 - \ln|x - y + 2|$$

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Question: Compute the Fourier integral of the function

$$f(x) = \begin{cases} \cos x & \text{if } |x| < \pi \\ 0 & \text{if } |x| \geq \pi \end{cases}$$

and deduce the values of

$$\int_0^\infty \lambda \frac{\sin(\lambda\pi)}{1-\lambda^2}\, d\lambda \quad \text{and} \quad \int_0^\infty \frac{\lambda \sin(2\lambda\pi)}{\lambda^2-1}\, d\lambda$$

(Hint: $2\cos a \cos b = \cos(a-b) + \cos(a+b)$)

Answer:

f is even on $(-\infty, \infty)$. So it has a Fourier cosine integral given by

$$f_c(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos(\lambda x)\, d\lambda$$

with $$A(\lambda) = 2 \int_0^\infty f(t) \cos(\lambda t)\, dt$$

$$A(\lambda) = 2 \int_0^\pi \cos t \cos \lambda t \, dt$$

$$= \int_0^\pi \left(\cos((1-\lambda)t) + \cos((1+\lambda)t) \right) dt$$

If $\lambda \neq 1$ then

$$A(\lambda) = \frac{1}{1-\lambda} \left[\sin(1-\lambda)t \right]_0^\pi + \frac{1}{1+\lambda} \left[\sin(1+\lambda)t \right]_0^\pi$$

$$= \frac{1}{1-\lambda} \sin(\pi - \lambda\pi) + \frac{1}{1+\lambda} \sin(\pi + \lambda\pi)$$

$$= \sin(\lambda\pi) \left(\frac{1}{1-\lambda} - \frac{1}{1+\lambda} \right)$$

$$= \frac{2\lambda \sin(\lambda\pi)}{1-\lambda^2}$$

Hence $$f_c(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2} \cos(\lambda x)\, d\lambda$$

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$\int_{-\infty}^{\infty} |f(x)|\,dx = \int_{-\pi}^{\pi} |\cos x|\,dx < \infty$

and f, f' are piecewise continuous on every finite interval.

f is continuous at $x=0$. Then

$f_c(0)$ converges to $f(0) = 1$

Hence $\frac{2}{\pi}\int_0^{\infty} \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2}\,d\lambda = 1$

Whence $\int_0^{\infty} \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2}\,d\lambda = \frac{\pi}{2}$

On the other hand, f is discontinuous at $x=\pi$, $f(\pi^-) = -1$ and $f(\pi^+) = 0$.

Then $f_c(\pi)$ converges to $\frac{-1+0}{2} = -\frac{1}{2}$

Hence $\frac{2}{\pi}\int_0^{\infty} \frac{\lambda \sin(\lambda\pi)}{1-\lambda^2}\cos(\lambda\pi)\,d\lambda = -\frac{1}{2}$

i.e., $\frac{1}{\pi}\int_0^{\infty} \frac{\lambda \sin(2\lambda\pi)}{1-\lambda^2}\,d\lambda = -\frac{1}{2}$

Whence $\int_0^{\infty} \frac{\lambda \sin(2\lambda\pi)}{\lambda^2-1}\,d\lambda = \frac{\pi}{2}$

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Question: let $f:\mathbb{R}\to\mathbb{R}$ be odd such that

$$f(x)=\begin{cases}1 & \text{if } 0<x<\pi\\ 0 & \text{if } x=\pi\end{cases}$$

and $f(x+2\pi)=f(x)$ for all x

Find the Fourier series $S_f(x)$ of f and deduce the value of

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

Answer: Since f is odd then

$$S_f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$=-\frac{2}{n\pi}\left[\cos nx\right]_0^{\pi}=\frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$=\begin{cases}0 & \text{if } n \text{ is even}\\ \frac{4}{n\pi} & \text{if } n \text{ is odd}\end{cases}$$

Hence $S_f(x)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\left((2k+1)x\right)$

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f and f' are piecewise continuous on $[-\pi, \pi]$.

f is continuous at $x = \frac{\pi}{2}$. Thus

$$S_f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right).$$ Hence

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left((2k+1)\frac{\pi}{2}\right) = 1$$

But $$\sin\left((2k+1)\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k$$

Therefore $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

Question: Let f be 2π periodic with

$$f(x) = \pi - |x| \quad \text{on } (-\pi, \pi).$$

Find the Fourier series of f and deduce the value of

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

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Since f is even then

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

with $$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx \quad \text{for } n \geq 1$$

$$a_0 = -\frac{1}{\pi}\left[(\pi - x)^2\right]_0^{\pi} = \pi.$$

For $n \geq 1$ $$\int_0^{\pi}(\pi - x)\cos nx\,dx = \left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx$$

$$= 0 - \frac{1}{n^2}\left[\cos nx\right]_0^{\pi} = \frac{1-(-1)^n}{n^2}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

Hence $$S_f(x) = \frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{4}{\pi(2k+1)^2}\cos((2k+1)x)$$

f and f' are piecewise continuous on $[-\pi, \pi]$.

f is continuous at $x = 0$. So

$$S_f(0) = f(0).$$

Hence $$\frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{4}{\pi(2k+1)^2} = \pi$$

whence $$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

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Solve $\begin{cases} e^{x-y}\ln(x)dx-(\ln(xy)-\ln y)dy=0 \\ \text{subject to} \quad y(1)=1 \end{cases}$

using exponential and logarithm properties, the DE reduces to:

$$e^{x}dx=e^{y}dy \Rightarrow \quad e^{x}=e^{y}+c$$

$$y(1)=1 \Rightarrow c=0 \quad \Rightarrow \quad e^{x}=e^{y} \quad \text{or} \quad \boxed{y=x}$$

Determine the largest region for which the following IVP has a unique solution:-

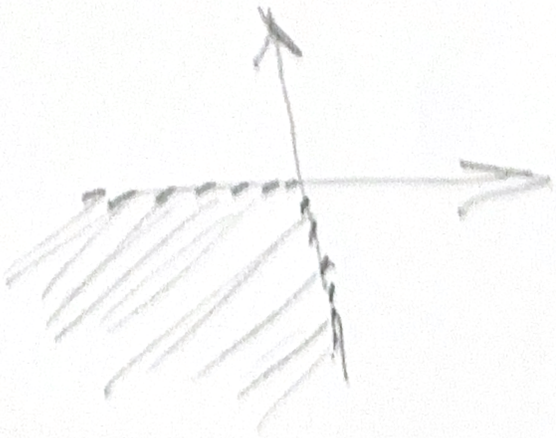
$$\begin{cases} \frac{dy}{dx}=\sqrt{\frac{y}{x}} \\ y(-1)=-2 \end{cases}$$

$f(x,y)=\sqrt{\frac{y}{x}}$, $\frac{\partial f}{\partial y}=\frac{1}{2x\sqrt{y/x}}$

so $x\neq 0$ and $\frac{y}{x}>0$

The initial point is $(-1,-2)$, so we are in the region $\{x<0, y<0\}$

$(-\infty,0)\times(-\infty,0)$



Question 1

Solve the differential equation:

$$(2xy + y^2)\,dx + x^2\,dy = 0.$$

Solution

Substitute $y = vx$ (Differential equation with homogeneous coefficients). Rewrite as

$$\frac{dy}{dx} = -\frac{2xy + y^2}{x^2} = -2\frac{y}{x} - \left(\frac{y}{x}\right)^2.$$

Then

$$v + x\frac{dv}{dx} = -2v - v^2 \quad \Longrightarrow \quad x\frac{dv}{dx} = -3v - v^2 = -v(3 + v).$$

Separate variables:

$$\frac{dv}{v(3+v)} = -\frac{dx}{x}.$$

Using partial fractions:

$$\frac{1}{v(3+v)} = \frac{1}{3}\left(\frac{1}{v} - \frac{1}{v+3}\right).$$

Integrate:

$$\frac{1}{3}\ln\left|\frac{v}{v+3}\right| = -\ln|x| + C \quad \Longrightarrow \quad \frac{v}{v+3} = Kx^{-3}, \quad K = e^{3C}.$$

Solve for v:

$$v = \frac{3K}{x^3 - K}$$

and hence

$$y = \frac{3Kx}{x^3 - K}.$$

Question 2

Solve the (IVP):

$$(e^{-x} - 2y)\,dx = dy \text{ with } y(0) = 2.$$

Solution

Rewrite as

$$\frac{dy}{dx} + 2y = e^{-x}.$$

This is a linear first order ODE with $P(x) = 2$ and $Q(x) = e^{-x}$. Integrating factor:

$$\mu(x) = e^{\int 2\,dx} = e^{2x}.$$

Then

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)\cdot Q(x)dx + C\right] = e^{-2x}\int e^{x}dx + Ce^{-2x} = e^{-x} + Ce^{-2x}.$$

$y(0) = 2 \quad \Longrightarrow \quad 2 = 1 + C \quad \Longrightarrow \quad C = 1.$

Finally the solution of the (IVP) is:

$$y = e^{-x} + e^{-2x}.$$

Question 3

Solve the differential equation:

$$(y + xy + 1)\,dy + y^2\,dx = 0, \text{ for } y > 0.$$

Solution

Here $M(x, y) = y^2$ and $N(x, y) = y + xy + 1$. Check for exactness:

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = y.$$

Since $2y \neq y$, the equation is not exact.

We seek an integrating factor μ that depends only on y.

$$\mu(y) = e^{\int \frac{N_y - M_x}{M} dy} = e^{\int \frac{-1}{y} dy} = e^{-\ln(y)} = \frac{1}{y}.$$

Multiply the differential equation by $\mu = 1/y$:

$$y\,dx + \left(1 + x + \frac{1}{y}\right)dy = 0.$$

Now this equation is exact. Let $F(x, y)$ be such that

$$\frac{\partial F}{\partial x} = y, \qquad \frac{\partial F}{\partial y} = 1 + x + \frac{1}{y}.$$

Integrate $\frac{\partial F}{\partial x} = y$ with respect to x:

$$F(x, y) = xy + \phi(y).$$

Then

$$\frac{\partial F}{\partial y} = x + \phi'(y) = 1 + x + \frac{1}{y} \quad \Longrightarrow \quad \phi'(y) = 1 + \frac{1}{y}.$$

Integrate:

$$\phi(y) = y + \ln y + C.$$

Hence

$$F(x, y) = xy + y + \ln y + C$$

Therefore the general solution is

$$xy + y + \ln y = C.$$

Q1 Write down the form of the particular solution y_p for the differential equation

$$y^{(4)} - y = (4-x)e^{x} + 6e^{-x} + \frac{1}{2}\cos x + e^{2x}\sin 3x.$$

Solution. The complementary solution is

$$y_c = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x.$$

Since,

$$f(x) = (4-x)e^{x} + 6e^{-x} + \frac{1}{2}\cos x + e^{2x}\sin 3x,$$

the particular solution must be

$$y_p = x(Ax+B)e^{x} + Cxe^{-x} + x\left(D\cos x + E\sin x\right) + e^{2x}\left(F\cos 3x + G\sin 3x\right).$$

Q2. Solve the differential equation

$$y'' - 3y' + 2y = 2x + 1 + 3xe^{x}.$$

Solution. We have

$$y_c = C_1 e^{x} + C_2 e^{2x}.$$

Here

$$f(x) = 2x + 1 + 3xe^{x}$$

and therefore,

$$y_p = (Ax+B) + x\left(Cx+D\right)e^{x}.$$

Thus, on substituting in the differential equation and comparing the coefficient, we get

$$A = 1,\ B = 2,\ C = -\frac{3}{2},\ D = -3.$$

Q3. Solve the differential equation

$$y'' + y = \frac{1}{1+\sin x}.$$

Solution. We have

$$y_c = C_1 \cos x + C_2 \sin x$$

and therefore,

$$y_p = u_1 \cos x + u_2 \sin x.$$

We find

$$W = 1, \quad W_1 = -\frac{-\sin x}{1+\sin x}, \quad W_2 = \frac{\cos x}{1+\sin x}$$

Therefore

$$u_1 = -\sec x + \tan x - x \quad \text{and} \quad u_2 = \ln\left(1+\sin x\right).$$

Q4. Solve the differential eq1uation

$$4x^2y^{''}+8xy^{'}+y=\sqrt{x}.$$

Solution. We find

$$y_c = C_1x^{-\frac{1}{2}}+C_2x^{-\frac{1}{2}}\ln x$$

and therefore,

$$y_p = u_1x^{-\frac{1}{2}}+u_2x^{-\frac{1}{2}}\ln x.$$

We find

$$W = x^{-2}, \quad W_1 = -\frac{1}{4}x^{-2}\ln x, \quad W_2 = \frac{1}{4}x^{-2}$$

and consequenly,

$$u_1= -\frac{1}{4}\left(x\ln x - x\right) \quad \text{and} \quad u_2=\frac{1}{4}x.$$

Question 1

Find a power series solution about $x = 0$ for the differential equation

$$y'' - xy' - y = 0$$

Solution

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$2a_2 - a_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) a_{k+2} - (k+1) a_k \right] x^k = 0$$

$$a_2 = \frac{a_0}{2},$$

$$a_{k+2} = \frac{a_k}{k+2}, \qquad k \geq 1,$$

For the even coefficients:

$$a_2 = \frac{a_0}{2}, \qquad a_4 = \frac{a_0}{2 \cdot 4}, \qquad a_6 = \frac{a_0}{2 \cdot 4 \cdot 6},$$

For the odd coefficients:

$$a_3 = \frac{a_1}{3}, \qquad a_5 = \frac{a_1}{3 \cdot 5}, \qquad a_7 = \frac{a_1}{3 \cdot 5 \cdot 7},$$

$$\boxed{y(x) = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots \right) + a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots \right)}$$

Question 2

Find a power series solution about $x = 0$ for the differential equation

$$y'' + xy = 0$$

$$y(0) = 1, \qquad y'(0) = 0.$$

Solution

Assume

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitute into the differential equation:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0.$$

From the constant term:

$$2a_2 = 0 \quad \Rightarrow \quad a_2 = 0.$$

$$a_{k+2} = -\frac{a_{k-1}}{(k+2)(k+1)}, \qquad k \geq 1$$

Using initial conditions:

$$a_0 = 1, \qquad a_1 = 0 \qquad a_2 = 0$$

$$a_3 = -\frac{a_0}{3 \cdot 2} = -\frac{1}{6} \qquad a_4 = -\frac{a_1}{4 \cdot 3} = 0, \qquad a_5 = -\frac{a_2}{5 \cdot 4} = 0, \qquad a_6 = -\frac{a_3}{6 \cdot 5} = \frac{1}{180}.$$

$$\boxed{y(x) = 1 - \frac{x^3}{6} + \frac{x^6}{180} + \cdots}$$