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Q1 a) $\int_0^\infty f^2 dx = \int_0^\infty f f \, dx \le \int_0^\infty |f||f| \, dx$

Since $|f| \le C \;\; \forall x \in [0,\infty)$, then

$\int_0^\infty f^2 dx \le C \int_0^\infty |f| dx$, and f is absolutely integrable, then $\int_0^\infty f^2 dx \le C \int_0^\infty |f| dx < \infty$

$\Rightarrow f \in \mathcal{L}^2(0,\infty)$.

Observe that the function $f(x) = \frac{1}{2x+1}$ is bounded on $[0,\infty)$, and $\frac{1}{2x+1} \in \mathcal{L}^2(0,\infty)$, in fact

$$\int_0^\infty \frac{1}{(2x+1)^2} dx = \frac{1}{2}$$

But $\frac{1}{2x+1}$ is not integrable on $(0,\infty)$, in fact

$$\int_0^\infty \frac{dx}{2x+1} = \frac{1}{2} \ln(2x+1) \Big|_0^\infty = \infty.$$

b) For $x \in (0,1)$, $\lim\limits_{n\to\infty} F_n(x) = \lim\limits_{n\to\infty} xn(1-x)^n = 0$

$F_n(0) = 0$, $F_n(1) = 0$

therefore $F_n \xrightarrow[n\to\infty]{} 0$ pointwise

$$\|F_n - 0\|^2 = n^2 \int_0^\infty x^2 (1-x)^{2n} dx = n^2 \left[\frac{-x^2 (1-x)^{2n+1}}{2n+1} \Big|_0^1 + 2 \int_0^1 \frac{x(1-x)^{2n+1}}{2n+1} dx \right]$$

$$= \frac{2n^2}{2n+1} \left[\frac{-x(1-x)^{2n+2}}{2n+1} \Big|_0^1 + \int_0^1 \frac{(1-x)^{2n+2}}{2n+2} dx \right]$$

$$= \frac{2n^2}{(2n+1)(2n+2)} \left[\frac{-(1-x)^{2n+3}}{2n+3} \Big|_0^1 \right] = \frac{2n^2}{(2n+1)(2n+2)(2n+3)}$$

$$\Rightarrow \lim_{n\to\infty} \|F_n - 0\|^2 = 0 \Rightarrow F_n \xrightarrow[n\to\infty]{} 0 \text{ in } \mathcal{L}^2(0,1)$$

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Q3 c) i) $\|f\|^2 = \int_0^\infty e^{2(1-5\lambda)x} dx = \frac{1}{2(1-5\lambda)} e^{2(1-5\lambda)x} \Big|_0^\infty$

$= \frac{1}{2(5\lambda-1)}$ for $1-5\lambda<0$ (i.e) $\lambda > \frac{1}{5}$

So $f \in \mathcal{L}^2(0,\infty)$ for $\lambda > \frac{1}{5}$

ii) $\|g\|^2 = \int_0^\infty e^{2\lambda x}(1-\cos x)dx = \frac{1}{2\lambda} e^{2\lambda x}(1-\cos x)\Big|_0^\infty - \int_0^\infty \frac{e^{2\lambda x}}{2\lambda} \sin x\, dx$

$= -\frac{1}{2\lambda}\int_0^\infty e^{2\lambda x} \sin x\, dx$ for $\lambda<0$

$= -\frac{1}{2\lambda}\left[\frac{1}{2\lambda} e^{2\lambda x}\sin x\Big|_0^\infty + \int_0^\infty \frac{1}{2\lambda} e^{2\lambda x}\cos x\, dx\right]$

$= -\frac{1}{4\lambda^2}\underbrace{\int_0^\infty e^{2\lambda x}\cos x\, dx}_{I}$ for $\lambda<0$

Then $-I = -\frac{1}{4\lambda^2} I - \int_0^\infty e^{2\lambda x} dx$

$\Rightarrow I\left(1-\frac{1}{4\lambda^2}\right) = \int_0^\infty e^{2\lambda x}dx = \frac{1}{2\lambda}e^{2\lambda x}\Big|_0^\infty = -\frac{1}{2\lambda}$ for $\lambda<0$

$\Rightarrow I(4\lambda^2-1) = -\frac{4\lambda^2}{2\lambda} = -2\lambda$

Then $I = \frac{2\lambda}{1-4\lambda^2}$ for $\lambda<0$

Hence $\|g\|^2 = -I + \int_0^\infty e^{2\lambda x}dx = \frac{2\lambda}{4\lambda^2-1} + \frac{1}{2\lambda}e^{2\lambda x}\Big|_0^\infty =$

$= \frac{2\lambda}{4\lambda^2-1} - \frac{1}{2\lambda}$ for $\lambda<0$

which means that $g \in \mathcal{L}^2(0,\infty)$ for $\lambda<0$.

iii) $\|h\|^2 = \int_0^\infty x e^{4\lambda x}dx = \frac{1}{4\lambda} x e^{4\lambda x}\Big|_0^\infty - \int_0^\infty \frac{e^{4\lambda x}}{4\lambda}dx = \frac{-1}{4\lambda}\int_0^\infty e^{4\lambda x}dx$ $(\lambda<0)$

$= -\frac{1}{(4\lambda)^2} e^{4\lambda x}\Big|_0^\infty = \frac{1}{16\lambda^2}$ for $\lambda<0$

Then $h \in \mathcal{L}^2(0,\infty)$ for $\lambda<0$

Q_2 a) : $\hat{f}(\xi) = \overbrace{2\int_0^\infty f(x)\cos(x\xi)\,dx}^{I}$ (Since f is even)

$$= 2\int_0^\infty e^{-ax}\cos(x\xi)\,dx$$

$$= 2\left[e^{-ax}\frac{\sin(x\xi)}{\xi}\Big|_0^\infty + a\int_0^\infty e^{-ax}\frac{\sin(x\xi)}{\xi}\,dx \right]$$

$$= \frac{2a}{\xi}\left[-e^{-ax}\frac{\cos(x\xi)}{\xi}\Big|_0^\infty - a\int_0^\infty e^{-ax}\frac{\cos(x\xi)}{\xi}\,dx \right]$$

$$= \frac{2a}{\xi^2} - \frac{a^2}{\xi^2}\underbrace{2\int_0^\infty e^{-ax}\cos(x\xi)\,dx}_{I}$$

$$\Rightarrow I\left(1+\frac{a^2}{\xi^2}\right) = \frac{2a}{\xi^2} \Rightarrow I = \hat{f}(\xi) = \frac{2a}{\xi^2+a^2}$$

or $f(x) = \frac{1}{\pi}\int_0^\infty \hat{f}(\xi)\cos(x\xi)\,d\xi$

$$\Rightarrow f(x) = \frac{2a}{\pi}\int_0^\infty \frac{\cos(x\xi)}{\xi^2+a^2}\,d\xi$$

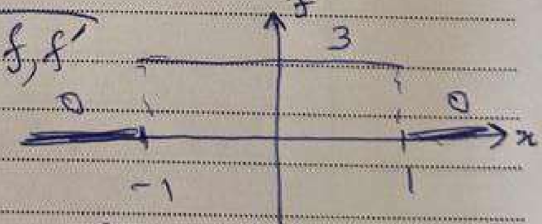
Let $x=0$, then $f(0) = 1 = \frac{2a}{\pi}\int_0^\infty \frac{d\xi}{\xi^2+a^2}$

$$\Rightarrow \int_0^\infty \frac{d\xi}{\xi^2+a^2} = \frac{\pi}{2a}$$

Since the integrand is even then

$$\int_{-\infty}^\infty \frac{d\xi}{\xi^2+a^2} = \frac{\pi}{a}.$$

Q_2 b) It is clear that f, f' are piecewise continuous on $(-\infty,\infty)$



and $\int_{-\infty}^\infty |f|\,dx = \int_{-1}^1 3\,dx = 6 < \infty$ (f is absolutely integrable)

and since f is even, then

$$f_c(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)\cos(\xi t)\right)\cos(x\xi)\,d\xi$$

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Since $\int_{-\infty}^{\infty} f(t)\cos(\xi t)dt = \int_{-1}^{1} 3\cos(\xi t)dt = 6\int_{0}^{1}\cos(\xi t)dt$

$= 6\,\frac{\sin(\xi t)}{\xi}\Big|_0^1 = \frac{6\sin\xi}{\xi}$

Thus $f_c(x) = \frac{6}{\pi}\int_0^{\infty}\frac{\sin\xi}{\xi}\cos(x\xi)\,d\xi$

Let $x=0$, then $f(0) = \frac{6}{\pi}\int_0^{\infty}\frac{\sin(\xi)}{\xi}d\xi = 3$

$\Rightarrow \int_0^{\infty}\frac{\sin\xi}{\xi}d\xi = \frac{\pi}{2} \Rightarrow \int_{-\infty}^{\infty}\frac{\sin\xi}{\xi}d\xi = \pi$

c) $\int_0^{\infty} f(x)\sin(x\xi)\,dx = \begin{cases} 2, & 0<\xi<1 \\ e^{-\xi}, & \xi>1 \end{cases}$

Since $B(\xi) = 2\int_0^{\infty} f(x)\sin(x\xi)\,dx$, when f is odd,

then $f(x) = \frac{1}{\pi}\int_0^{\infty} B(\xi)\sin(x\xi)\,d\xi$

$B(\xi) = 2\int_0^{\infty} f(x)\sin(x\xi)\,dx = \begin{cases} 4, & 0<\xi<1 \\ 2e^{-\xi}, & \xi>1 \end{cases}$

Thus $f(x) = \frac{4}{\pi}\int_0^1 \sin(x\xi)\,d\xi + \frac{2}{\pi}\underbrace{\int_1^{\infty} e^{-\xi}\sin(x\xi)\,d\xi}_{I}$

$= \frac{4}{\pi x}(1-\cos x) + \frac{2}{\pi} I$

$I = e^{-1}\left(\frac{\sin x + x\cos x}{x^2+1}\right)$

Hence $f(x) = \frac{4}{\pi x}(1-\cos x) + \frac{2}{\pi}e^{-1}\left(\frac{\sin x - x\cos x}{x^2+1}\right)$

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(Q_3 a) g and g' are piecewise continuous on $(-\pi,\pi)$
and $g(x+2\pi) = g(x)$, then g has a F.S

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Now $g(-x) = 1 - \frac{2}{\pi}|-x| = g(x) \Rightarrow g$ is even.

Thus $b_n = 0$, and

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2}{\pi}x\right)dx = \frac{2}{\pi}\left[x\Big|_0^{\pi} - \frac{x^2}{\pi}\Big|_0^{\pi}\right] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n}\Big|_0^{\pi}\right] \cancel{-\frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin(nx)}{n}\right]_0^{\pi}}$$

$$+ \frac{4}{\pi^2}\int_0^{\pi}\frac{\sin(nx)}{n}\,dx$$

$$= \frac{4}{\pi^2 n}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

Hence $g(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos(nx)$

$$= \frac{4}{\pi^2}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \dots + \frac{2}{(2n+1)^2}\cos(2n+1)x + \dots\right]$$

$$= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)}{(2n+1)^2}$$

Let $x=0$, then $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

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Q4 i) $\begin{cases} x^2u'' + xu' + \lambda u = 0 & x \in [1,2] \\ u(1)=0, \ u(2)=0 \end{cases}$ $\int \frac{x}{x^2} dx$

Multiply the DE by $\mu(x) = \frac{1}{x^2} e^{\int \frac{x}{x^2}dx} = \frac{1}{x}$

we get the DE $xu'' + u' + \frac{\lambda u}{x} = 0$

$$\Rightarrow \frac{d}{dx}\left(x \frac{du}{dx}\right) + \frac{\lambda}{x} u = 0 \longrightarrow (*)$$

Here $P(x) = x$, $r(x) = 0$, $w(x) = \frac{1}{x}$ (weight function)

Now multiply $(*)$ by u and integrate over $[1,2]$

$$\int_1^2 u \frac{d}{dx}\left(x \frac{du}{dx}\right) dx + \lambda \int_1^2 \frac{u^2}{x} dx = 0 \longrightarrow (**)$$

The first integral gives

$$\int_1^2 u \frac{d}{dx}\left(x \frac{du}{dx}\right) dx = u\, x \frac{du}{dx}\Big|_1^2 - \int_1^2 x(u')^2 dx$$

$$= -\int_1^2 x(u')^2 dx$$

Hence $(**)$ becomes

$$\lambda \int_1^2 \frac{u^2}{x} dx - \int_1^2 x(u')^2 dx = 0 \quad (***)$$

$$\Rightarrow \int_1^2 \left[\lambda \frac{u^2}{x} - x\left(\frac{du}{dx}\right)^2\right] dx = 0$$

From $(***)$: $\lambda = \dfrac{\int_1^2 x(u')^2 dx}{\int_1^2 \frac{u^2}{x} dx} > 0$

That is the eigenvalues are positive