King Saud University
Department of Mathematics

First Midterm Exam in Math 151, S1-1445H.

Calculators are not allowed

Q1. (a) Construct the truth table for $A = (p \wedge \neg q) \leftrightarrow (q \vee \neg r)$. [3]

(b) Without using truth tables, show that

$$(p \to q) \to r \equiv (\neg r \to p) \wedge (q \to r). \text{ [3]}$$

(c) Let n be an integer. Use a direct proof to show that if $3 \mid (n-1)$, then $9 \mid (n^2 + n + 16)$. [3]

Q2. (a) Use induction to show that $8 + 20 + 32 + \ldots + (12n - 4) = 6n^2 + 2n$ for all $n \geq 1$. [4]

(b) Let $\{u_n\}$ be a sequence defined by:

$$u_1 = 3, u_2 = 6, \text{ and } u_{n+1} = 2u_n - u_{n-1} + 2 \text{ for } n \geq 2.$$

Show that $u_n = n^2 + 2$ for all $n \geq 1$. [4]

Q3. (a) Let R be the relation from $A = \{-1, 0, 1, 2, 3\}$ to $B = \{1, 2, 3, 4, 5\}$ defined by:

$$aRb \Leftrightarrow a + 2b = 5.$$

(i) List all ordered pairs of R. [1]
(ii) Represent R with a matrix. [1]
(iii) Find the domain and image (range) of R. [1]

(b) Let $S = \{(a,b), (a,c), (a,d), (b,b), (b,c), (d,a), (d,d)\}$ and $T = \{(a,a), (a,d), (c,a), (c,c), (d,a)\}$ be relations on $E = \{a, b, c, d\}$.

(i) Find $T \circ S$. [2]
(ii) Find $\overline{S \cup T}$. [2
(iii) Find $S^{-1} - (S \cap T)$. [1]

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Q1) a) 9

p	q	r	$\neg q$	$\neg r$	$p\wedge\neg q$	$q\vee\neg r$	A.
T	T	T	F	F	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	T	F	F
T	F	F	T	T	T	T	T
F	T	T	F	F	F	T	F
F	T	F	F	T	F	T	F
F	F	T	T	F	F	F	T
F	F	F	T	T	F.	T	F.

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b)

$$(p\to q)\to r \equiv \neg(p\to q)\vee r$$
$$\equiv \neg(\neg p\vee q)\vee r$$
$$\equiv (p\wedge\neg q)\vee r.$$
$$\equiv (p\vee r)\wedge(\neg q\vee r)$$
$$\equiv (\neg(\neg r)\vee p)\wedge(\neg q\vee r)$$
$$\equiv (\neg r\to p)\wedge(q\to r).$$

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c) we assume that $3\mid(n-1)$

$\Rightarrow n-1=3q;\ q\in\mathbb{Z};$

$\Rightarrow n=3q+1 \Rightarrow n^2+n+16=(3q+1)^2+3q+1+16$

$=9q^2+6q+1+3q+1+16$

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$=9q^2+9q+18=9(q^2+q+2)$

$=9a;\ a=q^2+q+2\in\mathbb{Z}$

$\Rightarrow 9\mid(n^2+n+16)$

Q2) 8 a) $P(n): 8+20+32+\dots+(12n-4) = 6n^2+2n$; $\forall n \geq 1$.

BS: $P(1)$: $\left.\begin{array}{l} L.H.S = 8 \\ R.H.S = 6+2=8 \end{array}\right\} \Rightarrow P(1)$ is true.

I.S: Let $k \geq 1$, we prove that $P(k) \longrightarrow P(k+1)$.

we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$P(k): 8+20+32+\dots+(12k-4) = 6k^2+2k$;

we prove that: $P(k+1): 8+20+32+\dots+(12k+8) = 6(k+1)^2+2(k+1)$.

$8+20+32+\dots+(12k-4)+(12k+8) = 6k^2+2k+12k+8$

$= 6k^2+12k+6+2k+2$

$= 6(k^2+2k+1)+2(k+1)$

$= 6(k+1)^2+2(k+1)$

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$\Rightarrow P(k+1)$ is true $\Rightarrow \forall n \geq 1$ $P(n)$ is true.

b) $P(n): u_n = n^2+2$; $\forall n \geq 1$.

BS: $P(1): u_1 = 1+2 = 3$ True.

$P(2): u_2 = 4+2 = 6$ True.

I.S: Let $k \geq 2$, we assume that $P(1), P(2), \dots, P(k)$ are true and we prove that $P(k+1)$ is true.

$P(k+1): u_{k+1} = (k+1)^2+2$??

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$u_{k+1} = 2u_k - u_{k-1} + 2$

$\left.\begin{array}{l} P(k) \text{ true} \Rightarrow u_k = k^2+2, \\ P(k-1) \text{ true} \Rightarrow u_{k-1} = (k-1)^2+2 \end{array}\right\} \Rightarrow u_{k+1} = 2(k^2+2) - (k-1)^2 - 2 + 2$

$= 2k^2+4-k^2+2k-1-2+2$.

$= k^2+2k+1+2$

$= (k+1)^2+2$.

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1$; $P(n)$ is true.

P3) a) 8

(i) $R = \{(-1,3),(1,2),(3,1)\}$. 1

ii)

$$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$ 1

iii) domain(R) $= \{-1,1,3\}$.
image(R) $= \{1,2,3\}$. 1

b).

i) $T \circ S = \{(a,a),(a,c),(b,a),(b,c),(d,a),(d,d)\}$. 2

ii) $S \cup T = \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(c,a),(c,c),(d,a),(d,d)\}$.

$\overline{S \cup T} = \{(b,a),(b,d),(c,b),(c,d),(d,b),(d,c)\}$. 2

iii) $S^{-1} = \{(b,a),(c,a),(d,a),(b,b),(c,b),(a,d),(d,d)\}$.

$S \cap T = \{(a,d),(d,a)\}$.

$S^{-1} - (S \cap T) = \{(b,a),(c,a),(b,b),(c,b),(d,d)\}$. 1