King Saud University
Department of Mathematics

Final Exam in Math 151, S1-1445H.

(The exam is 2-pages long)

Calculators are not allowed

Q1. (a) Without using truth tables show that $(\neg p \to q) \to p \equiv q \to p$. (3)

(b) Use induction to show that $n^3 + 5n$ is divisible by 3 for all $n \geq 0$. (4)

(c) Suppose a and b are integers. Use contraposition to prove that if $a^2 - b^2 = 33$, then a is even or b is even. (3)

Q2. (a) Define a relation E on $\mathbb{Z} - \{0\}$ as mEn if and only if $3mn > 0$.

(i) Show that E is an equivalence relation. (3)

(ii) Find $[1]$ and $[-1]$. (2)

(b) Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 \geq y^2$.

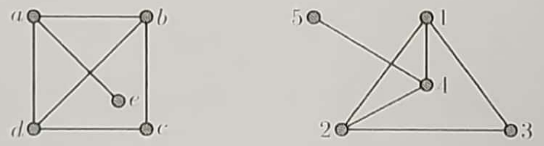
(i) Determine whether R is reflexive, symmetric, antisymmetric or transitive. (4)

(ii) Is R a partial ordering? (Justify your answer.) (1)

Q3. (a) Show that an undirected graph with degree sequence $4, 3, 2, 2, 1, x$ cannot be connected if it has exactly 6 edges. (2)

(b) Find the number of vertices n of the complete graph K_n, that has 36 edges. (2)

(c) Determine if the graphs G and H below are isomorphic. (Justify your answer.) (2)



G H

(d) Is the graph G in **(c)** bipartite? (Justify your answer.) (1)

(e) For the graph G in **(c)**, find a spanning tree with root a,

(i) using *depth-first* search; (1)

(ii) using *breadth-first* search. (1)

(f) Using alphabetical order, form a binary search tree for the words: *Mother, Father, Son, Daughter, Aunt, Uncle, Cousin.* (2)

Q4. (a) Prove the following Boolean identity:

$$\overline{x} + xy = \overline{x\overline{y}}. \text{ (2)}$$

(b) Let $f(x, y, z) = x(\overline{y} + x\overline{z})$ be a Boolean function.

(i) Find the complete sum-of-products expansion (CSP) of f. (2)

(ii) Find the complete product-of-sums expansion (CPS) of f. (2)

(c) Let $g(x, y, z) = xyz + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. (1)

(ii) Simplify g (i.e., write it in MSP form). (2)

10

P1) a) $(\neg p \to q) \to p \equiv (p \vee q) \to p$

$\equiv \neg(p \vee q) \vee p$

$\equiv (\neg p \wedge \neg q) \vee p$ 3

$\equiv (\neg p \vee p) \wedge (\neg q \vee p)$

$\equiv T \wedge (\neg q \vee p)$

$\equiv \neg q \vee p \equiv q \to p$.

b) $P(n): 3 \mid (n^3 + 5n), \forall n \geq 0$

B.S. $P(0): 3 \mid 0$ is true

4

I.S. Let $k \geq 0$; we assume that $P(k)$ is true and we prove that $P(k+1)$ is true

$P(k)$ true: $3 \mid k^3 + 5k \Rightarrow k^3 + 5k = 3a; a \in \mathbb{Z}$.

$P(k+1): 3 \mid ((k+1)^3 + 5(k+1))$

$(k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5$

$= k^3 + 5k + 3k^2 + 3k + 6$

$= 3a + 3k^2 + 3k + 6$

$= 3(a + k^2 + k + 2) = 3b;\ b = a + k^2 + k + 2 \in \mathbb{Z}$.

$\Rightarrow 3 \mid (k+1)^3 + 5(k+1)$

$\Rightarrow P(k+1)$ is true $\Rightarrow \forall n \geq 0\ 3 \mid (n^3 + 5n)$

c) by contraposition, we prove that if a is odd and b is odd then $a^2 - b^2 \neq 33$

a odd $\Rightarrow a = 2k+1;\ k \in \mathbb{Z}$

b odd $\Rightarrow b = 2t+1\ \ t \in \mathbb{Z}$

$a^2 - b^2 = (2k+1)^2 - (2t+1)^2$

$= 4k^2 + 4k + 1 - 4t^2 - 4t - 1$

$= 4k^2 + 4k - 4t^2 - 4t$ is even

3

and 33 is odd $\Rightarrow a^2 - b^2 \neq 33$.

1

P2) 10
a)
1). $m \in \mathbb{Z}-\{0\}$, $3m.m = 3m^2 > 0 \Rightarrow m \, E \, m$
$\Rightarrow E$ is reflexive

. $m, n \in \mathbb{Z}-\{0\}$ $m \, E \, n \Rightarrow 3mn > 0 \Rightarrow 3nm > 0$
$\Rightarrow n \, E \, m$
$\Rightarrow E$ is symmetric.

. $m, n, p \in \mathbb{Z}-\{0\}$; $m E n$; $n E p \Rightarrow \left.\begin{matrix} 3mn > 0 \\ 3np > 0 \end{matrix}\right\} \Rightarrow 9mn^2p > 0$
3
$\Rightarrow 3mp > 0$
$\Rightarrow m \, E \, p$
$\Rightarrow E$ is transitive
$\Rightarrow E$ is an equivalence relation on $\mathbb{Z}-\{0\}$.

ii) $[1] = \{m \in \mathbb{Z}\setminus\{0\};\ 3.m.1 > 0\}$
$= \{m \in \mathbb{Z}\setminus\{0\}$ st $m > 0\} \Rightarrow [1] = \mathbb{Z}^+$.
2
$[-1] = \{m \in \mathbb{Z}\setminus\{0\};\ 3.m(-1) > 0\}$
$= \{m \in \mathbb{Z}\setminus\{0\};\ m < 0\} = \mathbb{Z}^-$.

b).
(i) $x \in \mathbb{Z}$; $x^2 \geq x^2 \Rightarrow xRx \Rightarrow R$ is reflexive.
$x, y \in \mathbb{Z}$, $xRy \Rightarrow x^2 \geq y^2$ $\left(\begin{matrix} 2^2 \geq 1^2 \Rightarrow 2R1 \\ 1^2 \ngeq 2^2 \Rightarrow 1 \cancel{R} 2 \end{matrix} \right)$ 4
$\Rightarrow R$ is not symmetric.

. $1^2 \geq (-1)^2 \Rightarrow 1R(-1)$
$(-1)^2 \geq 1^2 \Rightarrow (-1)R(1) \Rightarrow$ R is not antisymmetric.
but $-1 \neq 1$

. $x, y, z \in \mathbb{Z}$; xRy and yRz. $x^2 \geq y^2$; $y^2 \geq z^2$
$\Rightarrow x^2 \geq z^2$
$\Rightarrow xRz$

$\Rightarrow R$ is transitive

ii) No; R is not antisymmetric.
1

2

Q3) 11

a) $4+3+2+2+1+x=12 \Rightarrow 12+x=12 \Rightarrow x=0$

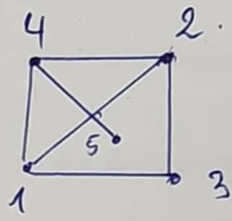
2 so the graph contain an isolated vertex

$\Rightarrow$ it is disconnected.

b) $|E(K_n)| = \frac{n(n-1)}{2} = 36 \Rightarrow n(n-1) = 72 \Rightarrow \boxed{n=9.}$ 2

c).

x	d	b	a	e	c
$f(x)$	1	2	4	5	3



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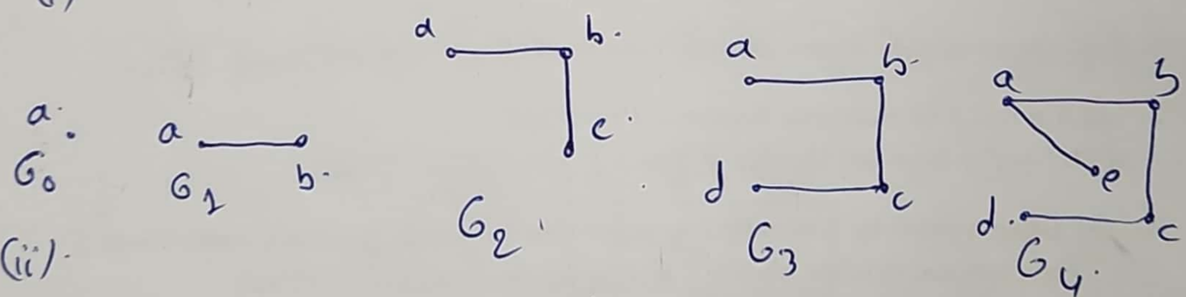
So G and H are isomorphic.

d). ~~mm~~ no because it contain a 3-cycle is the vertices a, b, d. 1

e).

(i)



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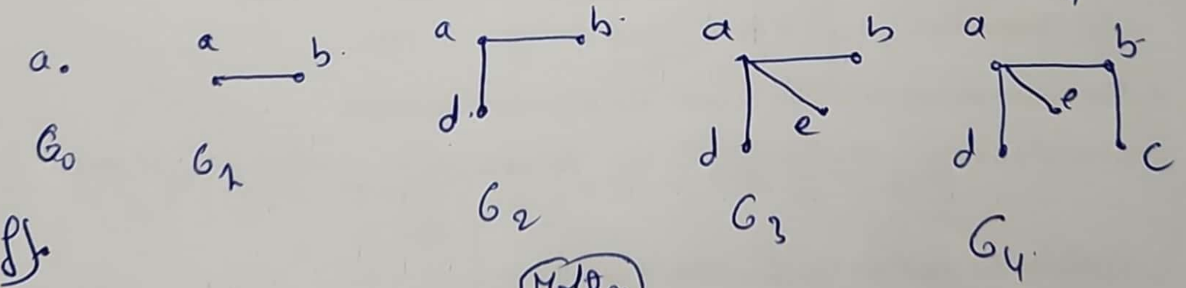
(ii)



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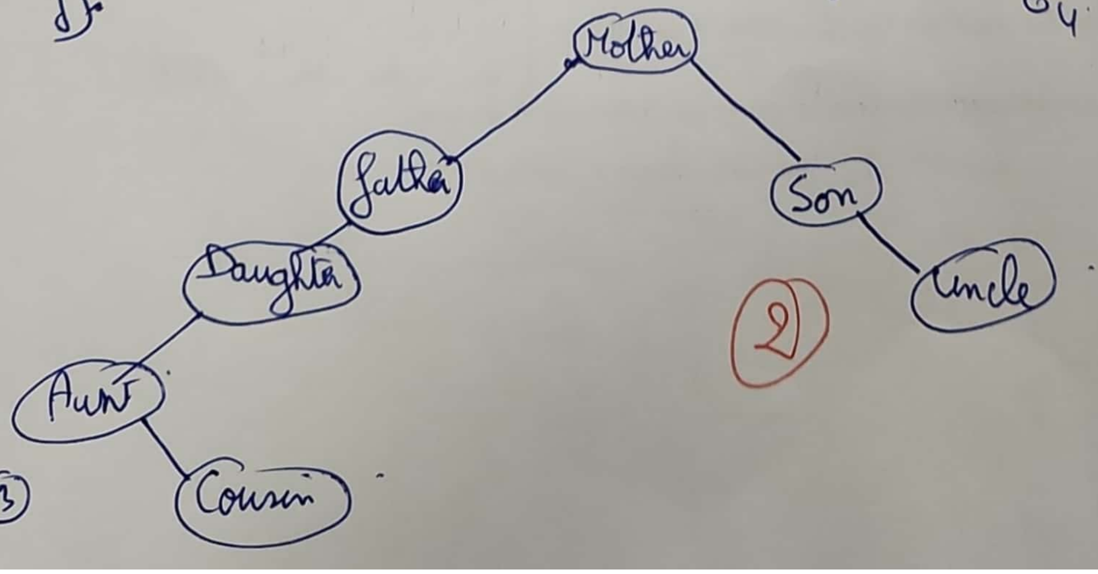
f).



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3

Q4) a) 3

$$\bar{x}+xy=(\bar{x}+x)\cdot(\bar{x}+y)$$
$$=1\cdot(\bar{x}+y)$$
$$=\bar{x}+y$$
$$=\overline{x\cdot\bar{y}}$$

2

b)

$f(x,y,z)=x\bar{y}+x\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	0	1	1	1
$\bar{x}$	0	0	0	0

$CSP(f)= xy\bar{z}+x\bar{y}\bar{z}+x\bar{y}z$

$CSP(\bar{f})= xyz+\bar{x}yz+\bar{x}y\bar{z}+\bar{x}\bar{y}z+\bar{x}\bar{y}\bar{z}$ 2

$CPS(f)=(\bar{x}+\bar{y}+\bar{z})(x+\bar{y}+\bar{z})(x+\bar{y}+z)(x+y+\bar{z})(x+y+z)$. 2

c)

i)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1		1	
$\bar{x}$		1	1	1

1

$MSP(g)= xyz+\bar{y}\bar{z}+\bar{x}\bar{z}+\bar{x}\bar{y}$. 2