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Research article

An approach based on the pseudospectral method for fractional telegraph equations

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Abstract: We aim to implement the pseudospectral method on fractional Telegraph equation. To implement this method, Chebyshev cardinal functions (CCFs) are considered bases. Introducing a matrix representation of the Caputo fractional derivative (CFD) via an indirect method and applying it via the pseudospectral method helps to reduce the desired problem to a system of algebraic equations. The proposed method is an effective and accurate numerical method such that its implementation is easy. Some examples are provided to confirm convergence analysis, effectiveness and accuracy.

Keywords: convergence analysis; Chebyshev cardinal functions; fractional Telegraph equation; pseudospectral method
Mathematics Subject Classification: 54A25, 65M70, 65Bxx, 35L20

1. Introduction

The objective of this project is to employ the pseudospectral approach to approximate the solution of the fractional Telegraph equation

$$\frac{\partial^\eta w(x,t)}{\partial t^\eta} + s_1 \frac{\partial^{\eta-1} w(x,t)}{\partial t^{\eta-1}} + s_2 w(x,t) = s_3 \frac{\partial^2 w(x,t)}{\partial x^2} + q(x,t), \tag{1.1}$$

subjected to boundary and initial conditions

$$w(0,t) = f_0(t), \quad w(1,t) = f_1(t), \quad t \in [0,1], \tag{1.2}$$

$$w(x,0) = p_0(x), \quad w'(x,1) = p_1(x), \quad x \in [0,1]. \tag{1.3}$$

Here, s_1, s_2 and s_3 are constants, and $q(x,t)$ is a known function. The fractional derivative is of the CFD type, so we will introduce it later.

As it is widely acknowledged, partial differential equations (PDEs) play a significant role in the simulation of multitudinous physical phenomena. Among them, the Telegraph equation, due to its application in modeling several phenomena such as electrical phenomena, signal processing and wave propagation, has been the focus of researchers. For this reason, solving it can be a fascinating challenge and attract the focus of scientists. Several methods are devoted to solving this equation. Interpolating scaling functions are applied to solve the problem via the collocation method [1]. Yang et al. [2] used orthogonal spline collocation method to solve the sub-diffusion equation. In [3], the authors proposed the collocation method for the desired equation using Chebyshev cardinal functions. Dehghan et al. [4] proposed a collocation method based on splines radial basis function. In [5], authors applied meshfree collocation method to solve Telegraph equation. The numerical results raised of the collocation method, taking the B-spline functions, are investigated in [6]. We list several numerical methods proposed to solve this equation, including shifted Jacobi collocation [7], multi-wavelet Galerkin method [8], Tchebyshev-Galerkin method [9], tau method [10] and so on.

In recent years, fractional calculations in modeling various physical phenomena have attracted the attention of engineers and specialists. Moreover, many types of equations with fractional derivatives have been considered, and many numerical schemes have been introduced to solve them, including the implementation of the Galerkin method for the fractional Riccati equation where biorthogonal Hermite cubic Spline is considered [11]. The application of the Tau method for a solution of space-time fractional PDEs has been considered based on interpolating scaling functions [12]. The multiwavelet method for solving the Cauchy-type problem is considered in [13]. In [14], the authors introduced the hybrid clique functions and applied them via the collocation method for fractional Schrödinger equation. A predictor-corrector compact difference scheme for a nonlinear fractional differential equation is applied in [15]. A nonlinear finite volume method is used to solve multi-term fractional sub-diffusion equation on polygonal meshes [16]. Zhang et al. [17] utilized collocation method based on Spline functions to solve the nonlinear fourth-order reaction sub-diffusion equation. In [18], the authors used the collocation method based on cubic B-spline functions to solve the time-fractional cable model. In [19], the authors applied a novel numerical technique to solve the time fractional reaction-diffusion model with a non-singular kernel, etc.

Among fractional PDEs, the fractional Telegraph equation, due to its application, has been considered as a challenging problem by many scientists. Hosseini et al. [20] applied a hybrid method based on finite differences and radial basis functions to solve the Eq (1.1). In [21], the spectral Galerkin method based on Legendre polynomials is studied for solving the problem. Mollahasani et al. [22] utilized a hybrid function scheme based on Block-Pulse-Functions and Legendre polynomials for solving (1.1). There is another form of time-fractional Telegraph equation

$$\frac{\partial^{2\eta} w(x,t)}{\partial t^{\eta}} + s_1 \frac{\partial^{\eta} w(x,t)}{\partial t^{\eta}} = s_2 \frac{\partial^2 w(x,t)}{\partial x^2} + q(x,t), \tag{1.4}$$

along with the initial conditions

$$w(x,0) = p_0(x), \quad w'(x,1) = p_1(x), \quad x \in [0,1], \tag{1.5}$$

and boundary conditions

$$w(1,t) + a_2 w_x(1,t) = f_1(t), \quad w(0,t) + a_1 w_x(0,t) = f_0(t), \quad t \in [0,1]. \tag{1.6}$$

For solving this equation, Saadatmandi et al. [23] used the Legendre polynomials and Tau method. In [24], the separable variable method is considered such as an analytical method to solve Eq (1.4). In [25], the authors solved (1.4) via the bireproducing kernel theorem. Akram et al. [19] used modified extended cubic B-spline functions to solve the Non-Linear Time-Fractional Telegraph Equation. The 2D time-fractional Telegraph equation is solved using modified fractional group iterative scheme [26].

The Chebyshev cardinal functions are applied as attractive bases for solving various kinds of equations. Owing to their abilities, they can be used in the pseudospectral and Galerkin methods. Recently, Shahriari et al. [27] studied the fractional Dirac problem using these bases. These bases are applied for solving the fractional Sturm-Liouville problem in [28]. Bin Jebreen et al. solved a family of time-fractional partial differential equation by these bases.

In this paper, for the first time, the Hyperbolic fractional Telegraph equation is solved using Cardinal Chebyshev functions. As you know, existing fractional derivatives and the nature of the hyperbolic partial differential equations are two factors that cause problems in the numerical solution. To overcome these issues, we apply the pseudospectral method based on Chebyshev cardinal functions. In this study, a matrix representation of the Caputo fractional derivative is introduced via an indirect method (using the relation between the Caputo fractional derivative and fractional integration) which plays a key role in our algorithm. An analysis of convergence is investigated to show the effectiveness and efficiency of the method.

The paper is organized as follows. The CCFs along with their properties are briefly described in Section 2. In Section 3, the pseudospectral method implements for approximating the solution of the problem. In this section, also, convergence analysis is investigated for the presented method. Some numerical examples are considered to give an affirmation of the method's efficiency.

2. Chebyshev cardinal functions (CCFs)

Consider the Chebyshev nodes as a set of numbers

$$\mathcal{Y} := \{y_j : T_{n+1}(y_j) = 0, j \in \Omega\}, \quad \Omega := \{1, 2, \dots, n+1\},$$

in which T_{n+1} is Tchebyshev polynomial of order $n+1$, n is a positive integer number, and $\{y_j\}_{j\in\Omega}$ are the roots of T_{n+1} on $[-1, 1]$. As we know, the roots of this polynomial are obtained via

$$y_j := \cos\left(\frac{2j-1}{2n+2}\pi\right), \quad \forall j \in \Omega. \tag{2.1}$$

It is so easy to verify that the Tchebyshev polynomials can be shifted on any arbitrary interval using a proper change of variables. The generated polynomials are known as the shifted Chebyshev polynomials, given by

$$T^*_{n+1}(x) := T_{n+1}\left(\frac{2(x-a)}{b-a} - 1\right), \quad x \in [a, b]. \tag{2.2}$$

Due to the change of variable $y = \left(\frac{2(x-a)}{b-a} - 1\right)$, the roots of the Chebyshev polynomials are also shifted to interval [a,b] and are given by $x_j = \frac{(y_j+1)(b-a)}{2} + a$.

The CCFs are the momentous type of cardinal functions that use orthogonal polynomials. These polynomials are defined via

$$\psi_j(x) = \frac{T^*_{n+1}(x)}{T^*_{n+1,x}(x_j)(x - x_j)}, \quad j \in \Omega, \tag{2.3}$$

in which the subscript x demonstrates differentiation with respect to x ($T^*_{n+1,x}(x_j) := \frac{d}{dx} T^*_{\omega+1}(x)|_{x=x_j}$). For simplicity and computational demands, these polynomials can be demonstrated as

$$\psi_j(x) = \rho \prod_{k=1, k\neq j}^{n+1} (x - x_k), \tag{2.4}$$

where $\rho = 2^{2n+1}/((b-a)^{n+1} T^*_{n+1,x}(x_j))$. These types of functions have significant property so that make them powerful tools for solving differential equations. According to Eq (2.3), It is obvious that

$$\psi_j(x_i) = \delta_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \quad i \in \Omega, \tag{2.5}$$

where δ_{ji} determines the Kronecker δ-function. This property is sufficient to demonstrate that any function $p(x)$ can be easily represented as an expansion based on Chebyshev cardinal functions, i.e.,

$$p(x) \approx p_n(x) = \sum_{j=1}^{n+1} p(x_j)\psi_j(x). \tag{2.6}$$

Let $\mathcal{D}$ be the derivative operator. Given $\omega \in \mathbb{N}$, the Sobolev space $H^\omega([0,1])$ is specified by

$$H^\omega([0,1]) = \{p \in L_2([0,1]) : \forall n' \leq \omega, \mathcal{D}^{n'} p \in L_2([0,1])\}.$$

This space is equipped with norm

$$\|p\|^2_{H^\omega([0,1])} = \sum_{k=0}^{\omega} \|p^{(k)}\|^2_{L_2([0,1])}, \tag{2.7}$$

and semi-norm

$$|p|^2_{H^{\omega,n}([0,1])} = \sum_{k=\min\{\omega,n\}}^{n} \|p^{(k)}\|^2_{L_2([0,1])}. \tag{2.8}$$

Lemma 1. *[29] Assuming the set of points $\{x_j\}_{j\in\Omega}$ as the shifted Gauss-Chebyshev points, we say that the bound of error for* (2.6) *can be approximated by*

$$\|p - p_n\|_{L_2([0,1])} \leq C_0 n^{-\omega} |p|_{H^{\omega,n}([0,1])}, \tag{2.9}$$

where $\omega \in \mathbb{N}$ and C_0 is a constant and independent of ω.

Putting the Chebyshev cardinal functions $\psi_j(x)$ into an $(n+1)$-dimensional vector, we introduce the vector function $\Psi(x)$ whose j-th entry is $\psi_j(x)$.

Lemma 2. *The derivative operator $\mathcal{D}$ can be represented by a square matrix D as*

$$\mathcal{D}(\Psi)(x) \approx D\Psi(x), \tag{2.10}$$

whose entries are given by

$$D_{j,i} = \mathcal{D}(\psi_j)(x_i) = \begin{cases} \sum\limits_{\substack{l=1\\ l\neq i}}^{n+1} \frac{1}{(x_i - x_l)}, & j = i, \\ \rho \prod\limits_{\substack{l=1\\ l\neq i,j}}^{n+1} (x_i - x_l), & j \neq i. \end{cases} \tag{2.11}$$

Proof. Using (2.6) and (2.10), we can easily verify that the elements of matrix D are computed by

$$D_{j,i} = \mathcal{D}(\psi_j)(x_i). \tag{2.12}$$

Motivated by (2.4), to derive the entries of matrix D, the following results can be obtained by taking the derivative with respect to the variable x from both sides of (2.4), viz,

$$\begin{aligned} \mathcal{D}(\psi_j)(x) &= \rho\mathcal{D} \prod_{\substack{k=1\\ k\neq j}}^{n+1} (x - x_k) = \rho \sum_{\substack{l=1\\ l\neq j}}^{n+1} \prod_{\substack{k=1\\ k\neq j,l}}^{n+1} (x - x_k) \\ &= \sum_{\substack{l=1\\ l\neq j}}^{n+1} \frac{T^*_{n+1}(x)}{(x - x_j)(x - x_l) T^*_{n+1,x}(x_j)} \\ &= \sum_{\substack{l=1\\ l\neq j}}^{n+1} \frac{1}{(x - x_l)} \psi_j(x). \end{aligned} \tag{2.13}$$

Thus, this gives rise to (2.11) and we have

$$\mathcal{D}(\psi_j)(x_i) = \sum_{\substack{l=1\\ l\neq i}}^{n+1} \frac{1}{(x_i - x_l)}, \quad i = j,$$

$$\mathcal{D}(\psi_j)(x_i) = \rho \prod_{\substack{l=1\\ l\neq i,j}}^{n+1} (x_i - x_l), \quad i \neq j.$$

□

The matrix D can be considered instead of the derivative from bases in the numerical method. This matrix is used to reduce and simplify the calculations. It is worthwhile to mention that when we utilize the spectral methods, it is no longer a need to find the derivative of the bases. Instead, we can use the matrix D.

Before introducing a matrix such as the operational matrix I_η for the fractional integral operator $\mathcal{I}_0^\eta$ of order $\eta > 0$, let us make some preliminaries about the fractional integral.

Definition 1. *Given $\eta \in \mathbb{R}^+$. Assuming the local integrable function $p : [0,1] \to \mathbb{R}$, the operator of the fractional integral $\mathcal{I}_0^\eta$ of order $\eta > 0$ is specified by*

$$\mathcal{I}_0^\eta(p)(x) := \frac{1}{\Gamma(\eta)} \int_0^x (x-z)^{\eta-1} p(z) dz, \quad x \in [0,1], \quad p \in L_1[0,1]. \tag{2.14}$$

With simple calculations, it can be shown that acting this operator on a power function is also a power function

$$\mathcal{I}_0^\eta(x^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\eta+1)} x^{\beta+\eta}. \tag{2.15}$$

Motivated by [30], the fractional integral can be bounded. To specify this bound, one can refer to the following Lemma.

Lemma 3. *The operator $\mathcal{I}_0^\eta$ is bounded in $L_p([0,1])$, i.e.,*

$$\|\mathcal{I}_0^\eta(p)\|_q \leq \frac{1}{\Gamma(\eta+1)} \|p\|_q, \quad 1 \leq q \leq \infty. \tag{2.16}$$

Lemma 4. *Given $\eta \in \mathbb{R}^+$, one can approximate acting the fractional integral operator on Ψ as follows*

$$\mathcal{I}_0^\eta(\Psi)(x) \approx I_\eta \Psi(x), \tag{2.17}$$

in which I_η is a square matrix of order $(n+1)$ whose entries are obtained by

$$[I_\eta]_{j,i} = \rho \sum_{k=0}^{n} z_{j,k} \frac{\Gamma(n-k+1)}{\Gamma(n-k+\eta+1)} x_i^{n-k+\eta}. \tag{2.18}$$

Proof. It is convenient to verify that

$$\prod_{\substack{k=1 \\ k \neq i}}^{n+1} (x - x_k) = \sum_{k=0}^{n} z_{i,k} x^{n-k}, \tag{2.19}$$

where

$$z_{i,0} = 1, z_{i,k} = \frac{1}{k} \sum_{l=0}^{k} c_{i,l} z_{i,k-l}, \ k = 1,2,\ldots,n, \ i = 1,2,\ldots,n+1,$$

and

$$c_{i,k} = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_j^k, \ k = 1,2,\ldots,n, \ i = 1,2,\ldots,n+1.$$

Using the aforementioned changes, Chebyshev cardinal functions can be rewritten as

$$\psi_j(x) = \rho \sum_{k=0}^{n} z_{j,k} x^{n-k}. \tag{2.20}$$

Putting (2.20) back into (2.14) and using Eq (2.15), one can write

$$\mathcal{I}_0^\eta \psi_j(x) = \rho \mathcal{I}_0^\eta (\sum_{k=0}^{n} z_{j,k} x^{n-k})$$

$$= \rho \sum_{k=0}^{n} z_{j,k} \mathcal{I}_0^{\eta}(x^{\eta-k})$$
$$= \rho \sum_{k=0}^{n} z_{j,k} \frac{\Gamma(n-k+1)}{\Gamma(n-k+\eta+1)} x^{n-k+\eta}.$$

This gives rise to reaching the desired result. □

2.1. Matrix representation of Caputo fractional derivative operator

Before introducing the operational matrix D_η for the fractional derivative operator ${}^c\mathcal{D}_0^\eta$, it is necessary to state some preliminaries about the CFD. Let $AC^\eta([0,1])$ is a space of functions such that

$$AC^\eta[0,1] = \{p : [0,1] \to \mathbb{C}, \quad \& \quad \mathcal{D}^{(\eta-1)}(p) \in AC[0,1]\}.$$

Assuming $p(x) \in AC^\eta[0,1]$, the Caputo fractional derivative, characterized by

$$({}^c\mathcal{D}_0^\eta p)(x) = \frac{1}{\Gamma(\kappa-\eta)} \int_0^x \frac{p^{(\kappa)}(t)dt}{(x-t)^{\eta-\kappa+1}} =: \mathcal{I}_0^{\kappa-\eta} \mathcal{D}^\kappa(p)(x), \tag{2.21}$$

exists for almost every $x \in [0,1]$. As a consequence of this definition, it is convenient to verify that [30]

$$({}^c\mathcal{D}_0^\eta (x)^{\alpha-1})(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)} x^{\alpha-\eta}, \quad (\alpha > \kappa). \tag{2.22}$$

Here, our objective is to introduce a square matrix D_η so that it satisfies

$${}^c\mathcal{D}_0^\eta(\Psi(x)) \approx D_\eta \Psi(x). \tag{2.23}$$

However, to gain the entries of the matrix D_η, we avoid finding them directly in a manner that is proposed for fractional integral. Instead, motivated by (2.21), matrix I_η can be used to obtain D_η, viz

$$\begin{aligned} {}^c\mathcal{D}_0^\eta(\Psi(x)) = \mathcal{I}_0^{\kappa-\eta} \mathcal{D}^\kappa(\Psi(x)) &\approx \mathcal{I}_0^{\kappa-\eta}(D^\kappa \Psi(x)) \\ &= D^\kappa \mathcal{I}_0^{\kappa-\eta}(\Psi(x)) \approx D^\kappa I_{\kappa-\eta}(\Psi(x)). \end{aligned}$$

Consequently, operational matrix D_η obtain by

$$D_\eta := D^\kappa I_{\kappa-\eta}. \tag{2.24}$$

Therefore, to obtain the operational matrix for the fractional derivative D_η, it is enough to obtain the operational matrix of the fractional integral of order $\kappa - \eta$ and multiply it by the κ power of the operational matrix of the derivative.

3. Method description

We emphasize that our objective is to approximate the solution of the Telegraph equation (1.1) by implementing the pseudospectral method. To develop the pseudospectral method for solving (1.1), the process begins by considering the unknown solution as an expansion based on CCFs

$$w(x,t) \approx w_n(x,t) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} W_{i,j} \Psi_i(x) \Psi_j(t) = \Psi^T(x) W \Psi(t), \tag{3.1}$$

in which W is a square matrix of order $n + 1$ whose elements should be found. Here and throughout the text, the superscript "T" denotes transpose.

Putting (3.1) back into (1.1) gives rise to

$$\frac{\partial^\eta w_n(x,t)}{\partial t^\eta} + s_1 \frac{\partial^{\eta-1} w_n(x,t)}{\partial t^{\eta-1}} + s_2 w_n(x,t) = s_3 \frac{\partial^2 w_n(x,t)}{\partial x^2} + q(x,t). \tag{3.2}$$

Using operational matrices D and D_η, one can introduce the residual in the approximation as

$$r(x,t) = \Psi^T(x)\left(WD_\eta + s_1 WD_{\eta-1} + s_2 W - s_3 D^{2^T} W - Q\right)\Psi(t), \tag{3.3}$$

where Q is an $n + 1$-dimensional matrix and obtain by

$$Q_{i,j} = q(x_i, x_j).$$

Let $WD_\eta + s_1 WD_{\eta-1} + s_2 W - s_3 D^{2^T} W = UW$ (this is always applicable), then Eq (3.3) may be rewritten as

$$r(x,t) = \Psi^T(x)\,(UW - Q)\,\Psi(t). \tag{3.4}$$

Pick distinct collocation points $\{x_j : j \in \Omega\}$, the pseudospectral method requires

$$r(x_i, x_j) = 0, \quad i, j \in \Omega. \tag{3.5}$$

This leads to specifying U as the solution of the linear system

$$UW = Q. \tag{3.6}$$

To solve this system, we convert U and Q to $\tilde{U}$ and $\tilde{Q}$, respectively. So we have a new system

$$A\tilde{U} = \tilde{Q}. \tag{3.7}$$

3.1. Convergence analysis

Considering p as a polynomial that interpolates the given sufficiently smooth function q at the points

$$q_{i,j} = q(x_i, t_j), \quad i, j = 1, 2, \ldots, n + 1,$$

the reminder formula is obtained by [31]

$$\begin{aligned} |q(x,t) - p(x,t)| = {} & \frac{\partial^n}{\partial x^n} q(\xi, t) \frac{\Pi_{i=1}^n (x - x_i)}{n!} + \frac{\partial^n}{\partial t^n} q(x, \tau) \frac{\Pi_{j=1}^n (t - t_j)}{n!} \\ & - \frac{\partial^{2n}}{\partial x^n t^n} q(\xi', \tau') \frac{\Pi_{i=1}^n (x - x_i) \Pi_{j=1}^n (t - t_j)}{n! n!}, \quad \tau, \xi, \tau', \xi' \in [0, 1]. \end{aligned} \tag{3.8}$$

Selecting the Chebyshev polynomials zeros as the interpolation nodes, (3.8) can be written as follows

$$|q(x,t) - p(x,t)| \le \left(\frac{1}{2}\right)^n \frac{1}{2^{n-1} n!} \sup_{\xi \in [0,1)} |\frac{\partial^n}{\partial x^n} q(\xi, t)| + \left(\frac{1}{2}\right)^n \frac{1}{2^{n-1} n!} \sup_{\tau \in [0,1)} |\frac{\partial^n}{\partial t^n} q(x, \tau)|$$

$$+\left(\frac{1}{2}\right)^{2n}\frac{1}{4^{n-1}(n!)^2}\sup_{\xi',\tau'\in[0,1)}|\frac{\partial^{2n}}{\partial x^n\partial t^n}q(\xi',\tau')|$$
$$\leq M_q\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\left(2+\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\right), \tag{3.9}$$

in which

$$M_q=\max\left\{\sup_{\xi\in[0,1)}|\frac{\partial^n}{\partial x^n}q(\xi,t)|,\ \sup_{\tau\in[0,1)}|\frac{\partial^n}{\partial t^n}q(x,\tau)|,\ \sup_{\xi',\tau'\in[0,1)}|\frac{\partial^{2n}}{\partial x^n\partial t^n}q(\xi',\tau')|\right\}.$$

Given $e=w-w_n$, subtracting (3.2) from

$$\frac{\partial^\eta w_n(x,t)}{\partial t^\eta}+s_1\frac{\partial^{\eta-1}w_n(x,t)}{\partial t^{\eta-1}}+s_2w_n(x,t)=s_3\frac{\partial^2 w_n(x,t)}{\partial x^2}+q_n(x,t),$$

the global error satisfies

$$\frac{\partial^\eta e(x,t)}{\partial t^\eta}+s_1\frac{\partial^{\eta-1}e(x,t)}{\partial t^{\eta-1}}+s_2e(x,t)=s_3\frac{\partial^2 e(x,t)}{\partial x^2}+q(x,t)-q_n(x,t). \tag{3.10}$$

Let the residual corresponding to (3.10) is

$$R_n(x,t)=\frac{\partial^\eta e(x,t)}{\partial t^\eta}+s_1\frac{\partial^{\eta-1}e(x,t)}{\partial t^{\eta-1}}+s_2e(x,t)-s_3\frac{\partial^2 e(x,t)}{\partial x^2}-q(x,t)+q_n(x,t). \tag{3.11}$$

Motivated by Theorem 2.2 [30], we have

$$|\frac{\partial^\eta}{\partial t^\eta}e(x,t)|=|\mathcal{I}_0^{\kappa-\eta}\frac{\partial^\kappa}{\partial t^\kappa}e(x,t)|\leq\frac{1}{\Gamma(\kappa-\eta)(\kappa-\eta+1)}||\frac{\partial^\kappa}{\partial t^\kappa}e(x,t)||_C,$$
$$|\frac{\partial^{\eta-1}}{\partial t^{\eta-1}}e(x,t)|=|\mathcal{I}_0^{\kappa-\eta}\frac{\partial^{\kappa-1}}{\partial t^{\kappa-1}}e(x,t)|\leq\frac{1}{\Gamma(\kappa-\eta)(\kappa-\eta+1)}||\frac{\partial^{\kappa-1}}{\partial t^{\kappa-1}}e(x,t)||_C.$$

Substituting these equations into (3.11) and using triangle inequality, we get

$$|R_n(x,t)|\leq|\frac{\partial^\eta e(x,t)}{\partial t^\eta}|+|s_1\frac{\partial^{\eta-1}e(x,t)}{\partial t^{\eta-1}}|+|s_2e(x,t)|+|s_3\frac{\partial^2 e(x,t)}{\partial x^2}|+|q(x,t)-q_n(x,t)|$$
$$\leq\frac{1}{\Gamma(\kappa-\eta)(\kappa-\eta+1)}\left(||\frac{\partial^\kappa}{\partial t^\kappa}e(x,t)||_C+|s_1|||\frac{\partial^{\kappa-1}}{\partial t^{\kappa-1}}e(x,t)||_C\right)$$
$$+|s_2|||e(x,t)||_C+|s_3|||\frac{\partial^2 e(x,t)}{\partial x^2}||_C+|q(x,t)-q_n(x,t)|.$$

To proceed, using (3.9), it can be found

$$|R_n(x,t)|\leq\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\left(2+\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\right)\left(\frac{M_{\frac{\partial^\kappa w}{\partial t^\kappa}}+|s_1|M_{\frac{\partial^{\kappa-1}w}{\partial t^{\kappa-1}}}}{\Gamma(\kappa-\eta)(\kappa-\eta+1)}\right.$$
$$\left.+|s_2|M_w+|s_3|M_{\frac{\partial^2 w}{\partial x^2}}+M_q\right). \tag{3.12}$$

Putting $C_y=\left(\frac{M_{\frac{\partial^\kappa w}{\partial t^\kappa}}+|s_1|M_{\frac{\partial^{\kappa-1}w}{\partial t^{\kappa-1}}}}{\Gamma(\kappa-\eta)(\kappa-\eta+1)}+|s_2|M_w+|s_3|M_{\frac{\partial^2 w}{\partial x^2}}+M_q\right)$ back into (3.12), we have

$$|R_n(x,t)|\leq C_y\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\left(2+\left(\frac{1}{2}\right)^{m}\frac{1}{2^{m-1}m!}\right). \tag{3.13}$$

Thus $|R_n(x,t)|\to 0$ as $m\to\infty$.

4. Illustrative examples

To demonstrate the performance of the proposed method, some examples are provided in this section. To illustrate the results and make a global view of the present method and its efficiency, sometimes, the absolute errors

$$e = |w(x_i, t_j) - w_n(x_i, t_j)|, \quad i.j = 1, \dots, n,$$

and L_2 error

$$L_2 - error = \left(\int_0^1 \int_0^1 |w(x,t) - w_n(x,t)|^2 dxdt \right)^{1/2},$$

are reported in tables or plotted in figures.

Example 1. *We dedicate the first example to the fractional Telegraph equation as*

$$\frac{\partial^\eta w(x,t)}{\partial t^\eta} + \frac{\partial^{\eta-1} w(x,t)}{\partial t^{\eta-1}} + w(x,t) = \pi \frac{\partial^2 w(x,t)}{\partial x^2} + q(x,t), \; 1 < \eta \leq 2,$$

with initial and boundary conditions

$$w(0,t) = 0, \; w(1,t) = t^3 \sin^2(1), \; w(x,0) = 0, \; w'(x,0) = 0,$$

in which

$$q(x,t) = \frac{6t^{3-\eta}\left(\sin^2(x)\right)}{\Gamma(4-\eta)} + \frac{6t^{4-\eta}\left(\sin^2(x)\right)}{\Gamma(5-\eta)} + t^3\left(\sin^2(x)\right) - \pi\left(2t^3\left(\cos^2(x)\right) - 2t^3\left(\sin^2(x)\right)\right).$$

The exact solution for this equation is given by $w(x,t) = t^3 \sin^2(x)$ [20].

To show the algorithm of the proposed method to solve this example, we describe it step by step here.

(1) Chose n;
(2) construct the Chebyshev cardinal functions of order n (refer to (2.3)*);*
(3) compute the Matrices D, I_η and D_η (refer to Lemma 2, Lemma 4 and (2.24)*, respectively);*
(4) approximate $w(x,t)$ using $w_n(x,t)$ (refer to (3.1)*);*
(5) put $w_n(x,t)$ back into (1.1) *(refer to* (3.2)*);*
(6) compute the residual $r(x,t)$ (refer to (3.4)*);*
(7) obtain the linear system (3.6) *using the shifted Chebyshev nodes $x_j = \frac{(y_j+1)}{2}$ ($j = 1, \dots, n$);*
(8) solve the linear system (3.8).

For instance, the coefficients' matrix in the obtained linear system for this example, taking $\eta = 1.75$ and $n = 3$, is equal to

$$\begin{bmatrix} 0.08931 & 0.0 & 0.0 & -0.33333 & 0.0 & 0.0 & 1.2440 & 0.0 & 0.0 \\ 0.0 & 0.08931 & 0.0 & 0.0 & -0.33333 & 0.0 & 0.0 & 1.2440 & 0.0 \\ 0.0 & 0.0 & 0.08931 & 0.0 & 0.0 & -0.33333 & 0.0 & 0.0 & 1.2440 \\ 0.0 & 0.0 & 0.0 & 0.08931 & -0.33333 & 1.2440 & 0.0 & 0.0 & 0.0 \\ 0.0 & -5.3324 & 0.0 & 5.5237 & 2.7602 & 3.3810 & 0.0 & -5.3324 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5121 & 5.3333 & -3.8215 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.08931 & -0.33333 & 1.2440 \\ 0.0 & 1.2440 & 0.0 & 0.0 & -0.33333 & 0.0 & 0.0 & 0.089316 & 0.0 \\ 0.0 & 0.0 & 1.2440 & 0.0 & 0.0 & -0.33333 & 0.0 & 0.0 & 0.089316 \end{bmatrix}.$$

Recall that the CFD of a function w tends to integer derivative as $\eta \to \kappa$. To demonstrate this effect, our results illustrated in Figure 1, obviously, demonstrate it. Figure 2 illustrates the effect of parameter n on L_2 error and confirm the convergence analysis. As you see, when n increases, the error decreases exponentially. Figure 3 demonstrates the approximate solution and corresponding absolute errors. Absolute error is reported for different values of x and t, taking $n = 9$ and $\eta = 1.75$, respectively, Table 1. It is worthwhile to mention that the process took a total of 77.390 *seconds of CPU time.*

Table 1. Absolute errors, taking $n = 9$ and $\eta = 1.75$ for Example 1.

x	t=0.2	t=0.4	t=0.6	t=0.8	t=1
0.1	$2.409e-09$	$1.711e-09$	$1.647e-09$	$1.903e-09$	$6.011e-10$
0.3	$7.426e-09$	$3.795e-09$	$2.722e-09$	$4.259e-09$	$8.402e-09$
0.5	$1.107e-08$	$4.569e-09$	$1.723e-09$	$1.055e-09$	$6.911e-10$
0.7	$1.033e-08$	$4.387e-09$	$1.637e-10$	$3.377e-09$	$1.876e-08$
0.9	$4.454e-09$	$1.497e-09$	$1.348e-09$	$4.402e-10$	$1.384e-08$

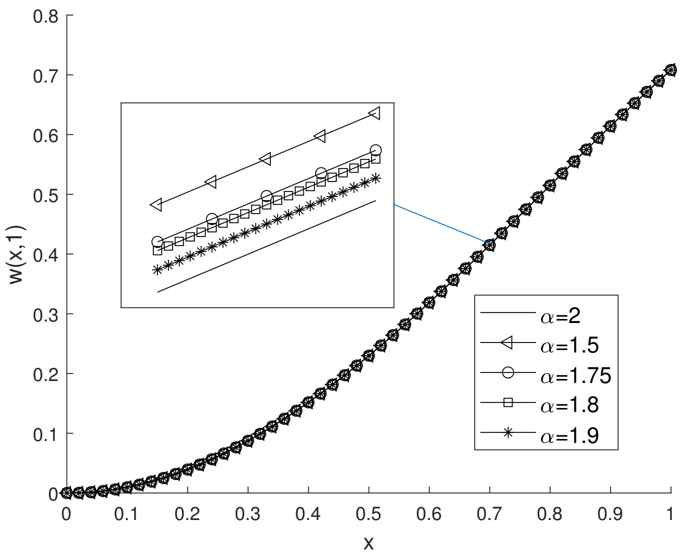

Figure 1. Approximate solution, taking different η for Example 1.

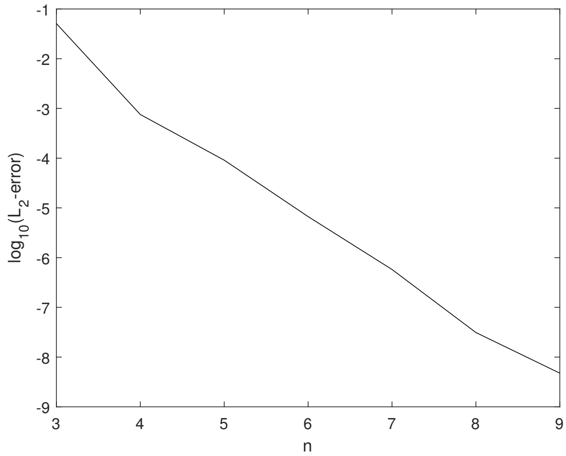

Figure 2. The L_2-error obtained by different number of n for Example 1.

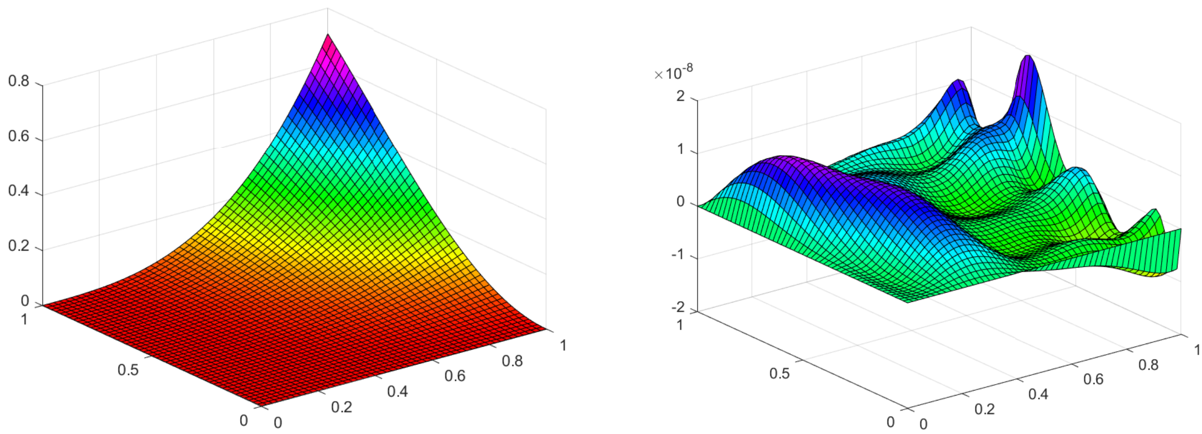

Figure 3. The plots of approximate and corresponding absolute error, taking $n = 9$ and $\eta = 1.75$, for Example 1.

Example 2. *We devote this example to the fractional Telegraph equation*

$$\frac{\partial^{\eta} w(x,t)}{\partial t^{\eta}} + \frac{\partial^{\eta-1} w(x,t)}{\partial t^{\eta-1}} + w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} + \left(t^2 - 2t + 2\right)\left(x - x^2\right)\mathrm{e}^{-t} + 2t^2\mathrm{e}^{-t}$$

with boundary and initial conditions

$$w(x,0) = 0,\ w'(x,0) = 0,\ w(0,t) = 0,\ w(1,t) = 0.$$

For this example, the exact solution is given by $w(x,t) = (x - x^2)e^{-t}t^2$ [4, 21, 22].

The approximate solution at $t = 0.1$, taking different values of η, is plotted in Figure 4. In Table 2, we observe a comparison between the present method and the method proposed in [22]. To demonstrate the accuracy and ability of the present method, the maximum absolute error at different times is *tabulated in Table 3. The approximate solution and corresponding absolute errors are illustrated in Figure 5. For more evidence of accuracy, Figure 6 is reported to show the L_2 errors with different choices of n for this example.*

Table 2. The comparison between hybrid functions method and the proposed method for Example 2.

	Proposed method		[22]	
t	$n = 6$	$n = 10$	$n = 6$	$n = 10$
0.6	5.21×10^{-5}	3.67×10^{-10}	4.43×10^{-3}	2.19×10^{-3}
0.7	4.39×10^{-5}	2.88×10^{-10}	2.67×10^{-3}	1.45×10^{-3}
0.8	2.92×10^{-5}	1.76×10^{-10}	2.14×10^{-3}	1.62×10^{-3}
CPU time	0.156	0.563	–	–

Table 3. The L_2 errors at different times with different choices of n, taking $\eta = 2$ for Example 2.

t	n=3	n=5	n=7	n=9	n=11
0.1	$9.67e-04$	$9.10e-05$	$9.10e-07$	$2.70e-09$	$5.18e-13$
0.3	$6.00e-03$	$3.77e-04$	$2.48e-06$	$6.47e-09$	$8.23e-13$
0.5	$1.06e-02$	$4.20e-04$	$2.84e-06$	$7.29e-09$	$3.27e-12$
0.7	$1.08e-02$	$2.97e-04$	$2.04e-06$	$5.23e-09$	$8.10e-12$
0.9	$4.60e-03$	$1.24e-04$	$6.41e-07$	$1.32e-09$	$1.74e-11$
CPU time	0.062	0.141	0.187	0.344	0.672

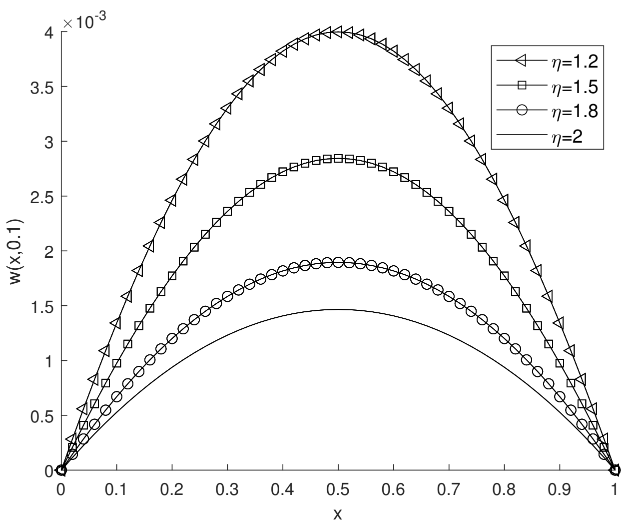

Figure 4. Approximate solution at $t = 0.1$, taking different η for Example 2.

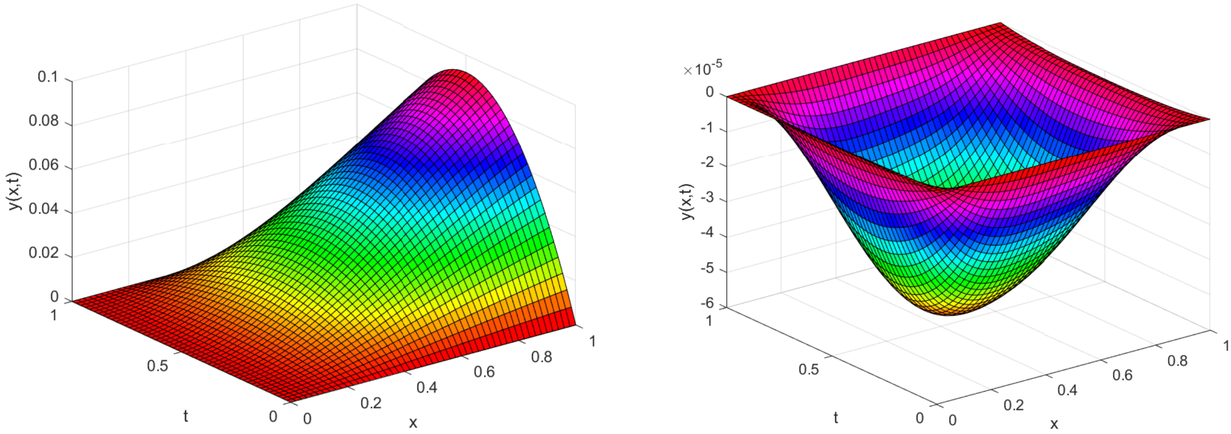

Figure 5. The plots of approximate and corresponding error, taking $n = 8$ and $\eta = 2$, for Example 2.

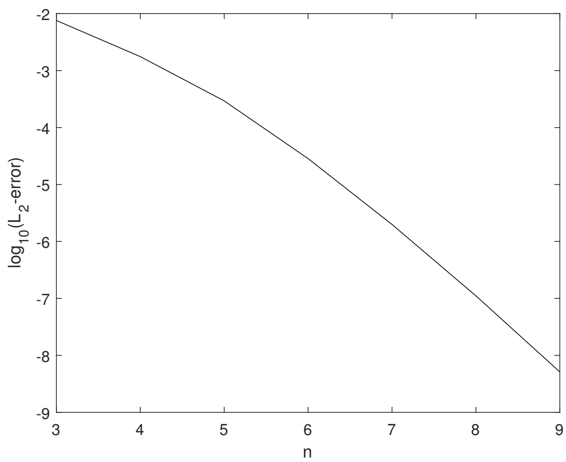

Figure 6. Effect of n on L_2 error, taking $n = 8$ and $\eta = 2$, for Example 2.

Example 3. *We devote this example to the fractional Telegraph equation*

$$\frac{\partial^\eta w(x,t)}{\partial t^\eta} + \frac{\partial^{\eta-1} w(x,t)}{\partial t^{\eta-1}} + w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} + \Gamma(\alpha+1)\sin(x) + \frac{\Gamma(\alpha+1)\, t^{\alpha-\beta}\sin(x)}{\Gamma(\alpha+1-\beta)} + 2t^\alpha \sin(x)$$

with boundary and initial conditions

$$w(x,0) = 0,\ w'(x,0) = 0,\ w(0,t) = 0,\ w(1,t) = \sin(1)t^\eta.$$

For this example, the exact solution is given by $w(x,t) = \sin(x)t^\eta$.

Table 4 shows the L_2-error at different times and different choices of η. In this table, the CPU time is also reported. It is clear that when the parameter n increases, the error decreases. The approximate solution and corresponding absolute errors are illustrated in Figure 7.

Table 4. L_2-error at different times for Example 3.

	$\eta = 1.75$		$\eta = 1.90$	
t	$n = 7$	$n = 10$	$n = 7$	$n = 10$
0.1	1.32×10^{-3}	7.14×10^{-4}	4.60×10^{-4}	2.94×10^{-4}
0.3	2.02×10^{-3}	9.89×10^{-4}	9.38×10^{-4}	2.53×10^{-4}
0.5	1.44×10^{-3}	6.75×10^{-4}	8.78×10^{-4}	4.35×10^{-4}
0.7	4.04×10^{-4}	1.77×10^{-4}	5.17×10^{-4}	2.08×10^{-4}
0.9	4.21×10^{-4}	2.27×10^{-4}	5.10×10^{-4}	3.21×10^{-4}
CPU time	2.624	5.215	2.765	6.002

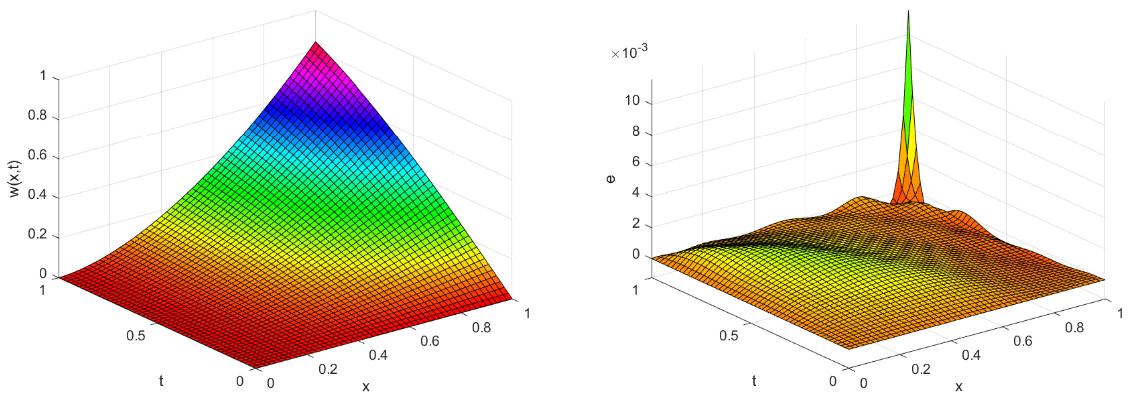

Figure 7. The plots of approximate and corresponding error, taking $n = 10$ and $\eta = 1.75$, for Example 3.

5. Conclusions

The pseudospectral method based on CCFs can be solved by the fractional Telegraph equation accurately. The presented method is easy to implement, and it solves problems of this type effectively and with appropriate accuracy. The convergence analysis also proves the method is convergent, and numerical examples confirm this investigation. Due to the cardinality property of the bases used, there is no need for integration to find the coefficients in the expansions, and this reduces the computational time and computational cost. For future work, we can use the pseudospectral method directly or apply the finite difference method and collocation method to solve the sophisticated models and the generalization of the method to two and three dimensions [32].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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