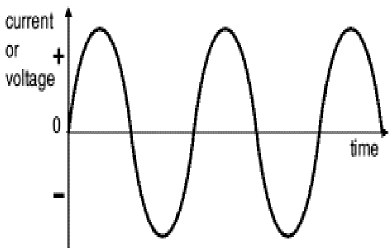
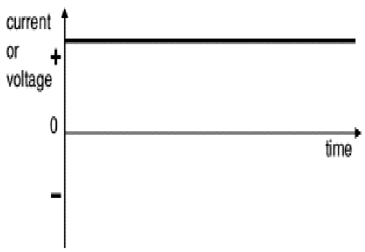
Chapter 7: Alternating-Current Circuits

7.1 Alternating-Current (AC) Sources

When we work with AC sources, we shall assume that the voltages, the currents, and the charges are all sinusoidal functions of time, with appropriate phases.



The voltage or current supplied by an AC source are sinusoidal with time.



The voltage or current supplied by an DC source are constant with time.

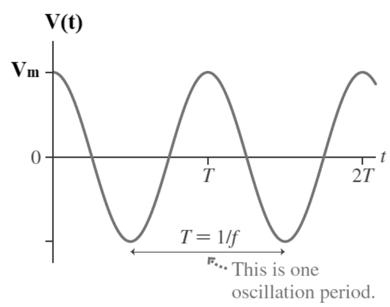
For instance, the time varying voltage from the AC source can be described by:

$$V = V_{max} \sin \omega t$$

where V_{max} is the peak voltage and $\omega = 2\pi f$ is the angular frequency, expressed in radians per second

$$\omega = 2\pi f = \frac{2\pi}{T}$$

and ***f*** is the frequency, expressed in Hertz.



Such voltage is said to be alternating and if it is applied to a circuit then the alternating current is of the form:

$$I = I_{max}\, sin(\omega\, t + \phi)$$

where I_{max} is the peak current and ϕ is the phase of the current with respect to the source voltage V.

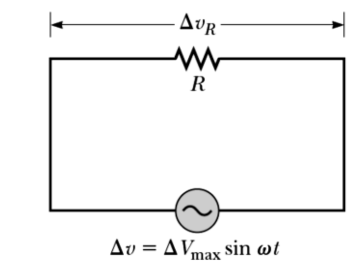
7.2 Resistors in an AC Circuit

The instantaneous current in the resistor shown in the circuit below is:

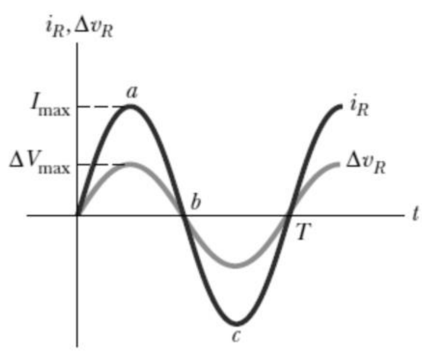
$$I = \frac{V_{\max}}{R} \sin \omega t = I_{\max} \sin \omega t$$

where

$$I_{\max} = \frac{V_{\max}}{R}$$



I and V both vary as (sin ωt) and reach their maximum, minimum and zero values at the same time, therefore, they are said to be in phase.



The rate at which energy is delivered to a resistor is the power, the instantaneous power is:

$$P = IV = I_{\max}^2 R \sin^2 \omega t$$

$$P_{avg} = I_{rms} V_{rms}$$

where

root mean square value $\quad I_{rms} = \dfrac{I_{\max}}{\sqrt{2}} = 0.707 I_{\max}$

$$V_{rms} = \frac{V_{\max}}{\sqrt{2}} = 0.707 I_{\max}$$

$$P_{avg} = \frac{1}{\pi} \int_0^{\pi} I_{max} V_{max} \sin^2 \omega t \, d(\omega t) = \frac{1}{2\pi} \int_0^{\pi} I_{max} V_{max} (1 - \cos 2\omega t) \, d(\omega t)$$

$$P_{avg} = \frac{I_{max} V_{max}}{2\pi} [\pi - 0] \Rightarrow P_{avg} = \frac{I_{max}}{\sqrt{2}} \frac{V_{max}}{\sqrt{2}} = I_{rms} V_{rms}$$

Example 7.1

The voltage output of an AC source is given by the expression:

$$V = 200 \sin \omega t$$

where V is in volts. Find the rms current in the circuit when this source is connected to a 100 Ω resistor.

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{V_{max}}{R} \right) = \frac{1}{\sqrt{2}} \left(\frac{200}{100} \right) = 1.41 \text{ A}$$

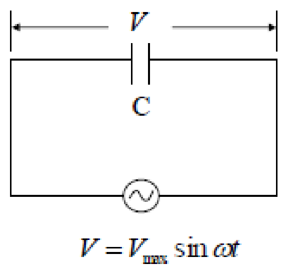
7.3 Capacitors in an AC Circuit

From Chapter 3, the charge on a capacitor is:

$$q=CV$$

If a capacitor is connected across an AC source as shown in the circuit below, then the instantaneous charge on it is:

$$q = CV_{max} \sin \omega t$$



The current in the circuit is:

$$I = \frac{dq}{dt} = \omega CV_{max} \cos \omega t$$

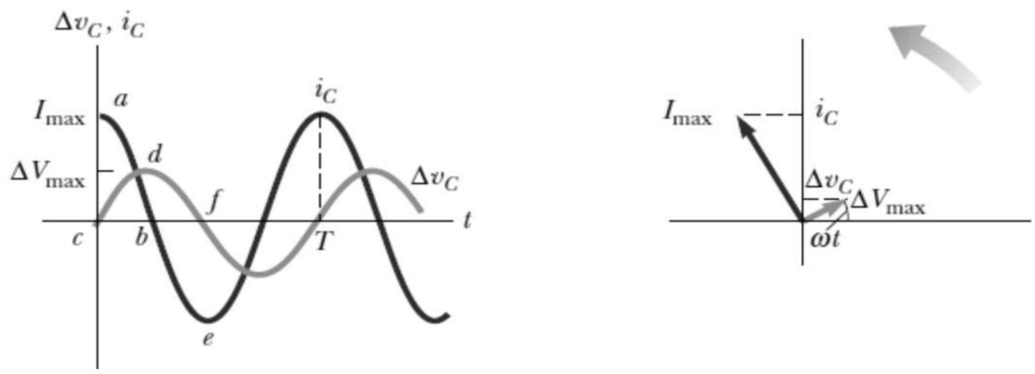
As $\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

Then,

$$I = \omega CV_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

This shows that the current is (π/2) out of phase with the voltage across the capacitor.

This means the current reaches its maximum value before the voltage reaches its maximum value by ¼ of a cycle as shown in the following plots.



The previous figure also shows the phasor diagram (on the right) which illustrate that the current always leads the voltage across a capacitor by π/2.

A phasor is a vector whose length is proportional to the maximum value of the voltage Vmax or the current Imax and it rotates counterclockwise at an angular speed equal to the angular frequency.

The projection of the phasor onto the vertical axis can gives the instantaneous value of the quantity it represents (voltage or current)

What will be the phasor diagram of a resistor connected across an AC source?

From the previous equation:

$$I_{max} = \omega C V_{max} = \frac{V_{max}}{X_C}$$

where X_c is defined as the capacitive reactance with unites of ohms and it varies with frequency

$$X_C = \frac{1}{\omega C}$$

And the instantaneous power in a purely capacitive AC circuit is:

$$P = VI = \frac{1}{2}\omega C V_{max}^2 \sin 2\omega t$$

Example 7.2

An 8-μF capacitor is connected to the terminals of a 60-Hz AC source whose rms voltage is 150 V. Find the capacitive reactance and the rms current in the circuit.

$$X_c = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

$$X_c = \frac{1}{2\pi(60)(8\times10^{-6})} = 332\ \Omega$$

$$I_{rms} = \frac{V_{rms}}{X_c} = \frac{150}{332} = 0.452\ A$$

Example 7.3

The voltage output of an AC source is given by the expression

$$V=100 \sin 1000\, t$$

where V is in volts. If this source is connected to a 2μF capacitor, find the following:

a) Capacitive reactance.
b) The instantaneous current.
c) The instantaneous charge on the capacitor.
d) The instantaneous power in the circuit.

a) $$X_c = \frac{1}{\omega C} = \frac{1}{1000(2\times10^{-6})} = 500\ \Omega$$

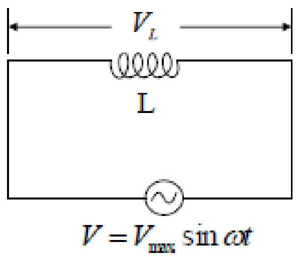
b) $$I = \frac{100}{500}\sin\left(1000t+\frac{\pi}{2}\right) = 0.2\sin\left(1000t+\frac{\pi}{2}\right)\ A$$

c) $$q = CV_{max}\sin 1000t = 2\times10^{-4}\sin(1000t)\ C$$

d) $$P = \frac{1}{2}\omega C V_{max}^2 \sin 2\omega t = 10\sin(2000t)\ W$$

7.4 Inductors in an AC Circuit

If an AC circuit consisting of an inductor of inductance L as shown in the figure below, we can apply Kirchhoff's second loop rule to this circuit:



$$V + V_L = 0$$

$$V - L\frac{dI}{dt} = 0 \;\; \Rightarrow \;\; L\frac{dI}{dt} = V_{max}\, sin(\omega\, t)$$

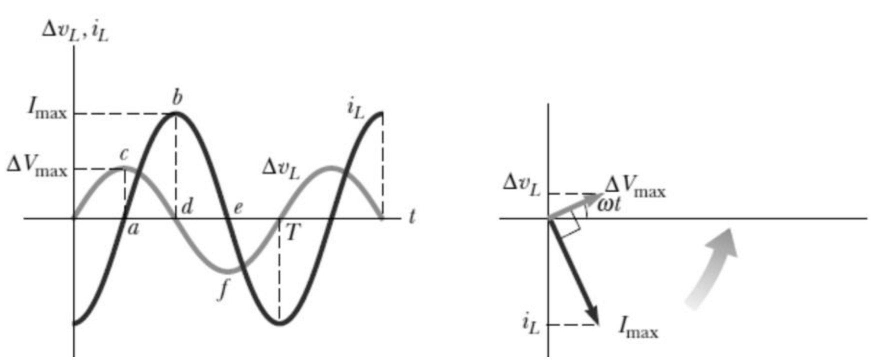
$$dI = \frac{V_{max}}{L} sin(\omega\, t) dt$$

$$I_L = \frac{V_{max}}{L}\int sin(\omega\, t) dt = -\frac{V_{max}}{\omega L} cos(\omega\, t)$$

$$\Rightarrow \;\; I_L = \frac{V_{max}}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right)$$

This shows that the the instantaneous current I_L in the inductor and the instantaneous voltage V_L across the inductor are out of phase.

This means the voltage reaches its maximum value ¼ of a period before the current reaches its maximum value as shown in the following plots.



From the last equation:

$$I_{\max} = \frac{V_{\max}}{\omega L} = \frac{V_{\max}}{X_L}$$

where X_L is defined as the inductive reactance with unites of ohms and it varies with frequency

where,

$$X_L = \omega L$$

And the instantaneous power in a purely inductive AC circuit is:

$$P = VI = \frac{1}{2}\omega L I_{\max}^2 \sin 2\omega t = V_{rms} I_{rms} \sin 2\omega t$$

Example 7.4

In a purely inductive AC circuit, L = 25 mH and the rms voltage is 150 V. Calculate the inductive reactance and rms current in the circuit if the frequency is 60 Hz.

$$X_L = \omega L = 2\pi f L = 2\pi(60)(25\times10^{-3})$$

$$X_L = 9.42\ \Omega$$

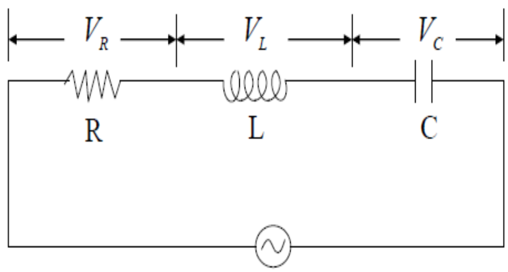
$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{150}{9.42} = 15.9\ A$$

7.5 The RLC Series Circuit

The following circuit contains a resistor, an inductor, and a capacitor connected in series across an AC voltage source.

As shown previously, the instantaneous applied voltage is given by:

$$V = V_{max} \sin \omega t$$



And the current in this case is given by the following equation:

$$I = I_{max} \sin(\omega t - \alpha)$$

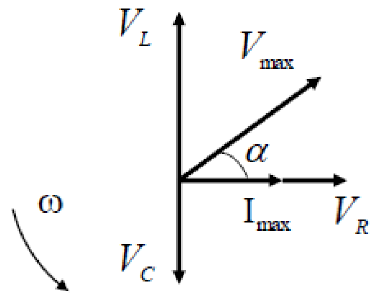
where α is the phase angle between the current and the voltage.

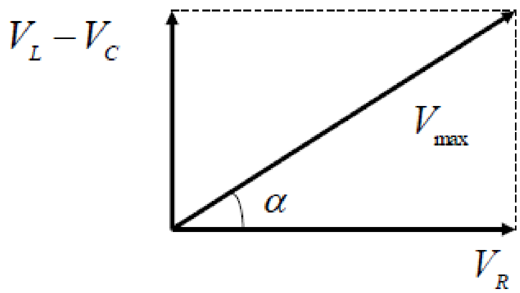
As the circuit elements in this circuit are connected in series, the current must be the same at any instant (with the same amplitude and phase) in each element. However, the voltage across each element has a different amplitude and phase as discussed int the previous sections:

- The voltage across the resistor is in phase with the current.
- The voltage across the inductor leads the current by π/2.
- The voltage across the capacitor lags behind the current by π/2.

The following figure shows the phasor diagram of the instantaneous voltages across the three elements of the series RLC circuit.

The figure illustrates that the inductance and capacitance phasors are added together and then added vectorially to the resistance phasor.





This figure illustrates that the inductance and capacitance phasors are added together and then added vectorially to the resistance phasor.

It also shows that the total voltage V_{max} makes an angle α (the phase angle) with I_{max}.

Therefore, V_{max} can be expressed as:

$$V_{max} = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(I_{max} R)^2 + (I_{max} X_L - I_{max} X_C)^2}$$

where

$$V_{max} = I_{max} \sqrt{R^2 + (X_L - X_C)^2}$$

$$V_{max} = I_{max} Z$$

Where Z is called the impedance of the circuit and its unit is ohm

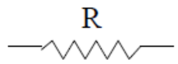
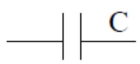
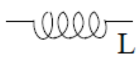
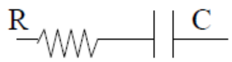
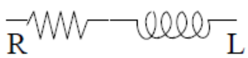
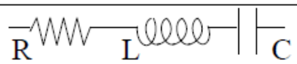
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The impedance (and the current) in an AC circuit depend on the resistance, the inductance, the capacitance, and the frequency (because X_L and X_C are frequency dependent).

From the right triangle in the phasor diagram in the previous figure, the phase angle α is found as:

$$\alpha = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

The table below shows impedance values and phase angles for various circuit-element combinations

Circuit Elements	Impedance Z	Phase Angle
	R	0^0
	X_C	-90°
	X_L	$+90^0$
	$\sqrt{R^2+X_C^2}$	Negative, between -90° and 0°
	$\sqrt{R^2+X_L^2}$	Positive, Between 0° and 90°
	$\sqrt{R^2+(X_L-X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

The average power delivered to the series RLC circuit is given by:

$$P_{avg}=I_{rms}V_{rms}\cos\alpha$$

where the quantity (**cos α**) is called the power factor

Example 7.5

A series RLC circuit has $R = 425\ \Omega$, $L = 1.25$ H, and $C = 3.5\ \mu F$. It is connected to an AC source with $f = 60$ Hz and $V_{max} = 150$ V.

a) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit.
b) Find the maximum current in the circuit.
c) Find the phase angle between the current and voltage.
d) Find the maximum voltage across each element.
e) Calculate the average power delivered to the series RLC circuit

a) $X_L = wL = 2\pi f L = 2\pi(60)(1.25) = 471.2\ \Omega$

$X_c = \frac{1}{wc} = \frac{1}{2\pi f C} = \frac{1}{2\pi(60)(3.5\times10^{-6})} = 757.9\ \Omega$

$Z = \sqrt{R^2 + (X_L - X_c)^2} = \sqrt{(425)^2 + (471.2 - 757.9)^2}$

$Z = 512.7\ \Omega$

b) $I_{max} = \frac{V_{max}}{Z} = \frac{150}{512.7} = 0.293\ A$

c) $\alpha = \tan^{-1}\left(\frac{X_L - X_c}{R}\right) = \tan^{-1}\left(\frac{471.2 - 757.9}{425}\right)$

$\alpha = -34^\circ$

d) $V_R = I_{max} R = 0.293\,(425) = 124.5\ V$

$V_L = I_{max} X_L = 0.293\,(471.2) = 138\ V$

$V_c = I_{max} X_c = 0.293\,(757.9) = 222\ V$

e) $P_{avg} = I_{rms} V_{rms} \cos\alpha = \frac{I_{max}}{\sqrt{2}} \frac{V_{max}}{\sqrt{2}} \cos\alpha$

$P_{avg} = \frac{(0.293)(150)}{2} \cos(-34) = 18.2\ W$

7.6 Resonance in a Series RLC Circuit

A series RLC circuit which can be an electrical oscillating system becomes in resonance when the driving frequency allows I_{rms} to have its maximum value. The rms current can be written as:

$$I_{rms} = \frac{V_{rms}}{Z}$$

$$I_{rms} = \frac{V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

I_{rms} has its maximum value when

$$X_L = X_C$$

$$\omega_r L = \frac{1}{\omega_r C}$$

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

The resonance frequency:

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

A radio's receiving circuit is an important application of a resonant circuit. The radio is tuned to receive a signal of a specific frequency by varying a capacitor that is an element of the receiving circuit.

Example 7.6

Consider a series RLC circuit for which R = 150 Ω, L = 20 mH, V_{rms} = 20 V, and ω =5000 s^{-1}. Determine the value of the capacitance for which the current is a maximum.

$$\omega_r = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{\omega_r^2 L} = \frac{1}{(5000)^2 (20 \times 10^{-3})}$$

$$\Rightarrow \quad C = 2 \mu F$$