King Saud University
College of Sciences
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Solution of the second midterm exam ACTU–464-474 (25%)

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Problem 1.

Let $(X_k)_{k=1}^n$ be a finite family of independent random variables (losses) such that X_k is exponentially distributed with parameter $\lambda_k > 0$). If the moment generating functions of $S = \sum_{k=1}^n X_k$ is of the from

$$M_S(t) = \sum_{k=1}^{n} a_k \frac{\lambda_k}{\lambda_k - t} \text{ for all } t < \min_{1\le k\le n}(\lambda_k) \tag{0.1}$$

where $(a_k)_{k=1}^n$ are real numbers such that $\sum_{k=1}^n a_k = 1$.
Then the probability density function of S is given by

$$f_S(x) = \sum_{k=1}^{n} a_k \lambda_k e^{-\lambda_k x} \text{ for all } x > 0.$$

Consider three independent losses ($n = 3$) X_1, X_2, X_3. For $k = 1, 2, 3$, X_k has an exponential distribution with mean $\frac{1}{k}$. Denote by S the aggregate loss.

1. Find the **M**oment **G**enerating **F**unction of S.
2. Decompose **MGF** of S into the form (0.1) and find the density f_S of S.
3. Compare the derivative of $(1 - e^{-x})^3$ with $f_S(x)$. Calculate the 90% premium of the aggregate loss.

Solution:

1. We know that

$$M_{X_1+X_2+X_3}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t) = \frac{1}{1-t} \cdot \frac{2}{2-t} \cdot \frac{3}{3-t}$$

2. By decomposition of fractions we have

$$\frac{1}{1-t} \cdot \frac{2}{2-t} \cdot \frac{3}{3-t} = 3\frac{1}{1-t} - 3\frac{2}{2-t} + \frac{3}{3-t},$$

therefore thanks to (0.1) we can write

$$f_S(x) = 3e^{-x} - 3\left(2e^{-2x}\right) + e^{-3x}.$$

3. We observe that
$$\left(\left(1-e^{-x}\right)^3\right)' = 3e^{-x}\left(1-e^{-x}\right)^2 = f_S(x),$$
hence $F_S(x) = \left(1-e^{-x}\right)^3$ is the c.d.f. of S. The requested 90% premium of S is given by the solution of the equation
$$\mathrm{P}\left(S \le P_{0.9}\right) = 0.9 = F_S(P_{0.9})$$
that is $\left(1-e^{-P_{0.9}}\right)^3 = 0.90$, that is $1-(0.90)^{\frac{1}{3}} = e^{-P_{0.9}}$ and then
$$P_{0.9} = -\ln\left(1-(0.90)^{\frac{1}{3}}\right) = \mathbf{3.3665}.$$

Problem 2. The random variables X_1, X_2 , X_3 and X_4 are independent with probability mass functions $f_1(k)$, $f_2(k)$, $f_3(k)$ and $f_4(k)$, respectively. Set $S = X_1 + X_2 + X_3 + X_4$. Your are given:

k	$f_1(k)$	$f_2(k)$	$f_3(k)$	$f_4(k)$
0	0.5	0.4	0.4	0.3
1	0.5	0.3	0.3	0.3
2	0	0.3	0.2	0.4
3	0	0	0.1	0

1. Complete the following table

k	$f_1(k)$	$f_1 * f_2(k)$	$f_1 * f_2 * f_3(k)$	$f_1 * f_2 * f_3 * f_4(k)$
0	0.5	?	0.080	0.024
1	0.5	0.35	?	0.084
2	0	0.30	0.265	?

2. What is $\mathrm{P}\left(S \ge 3\right)$?

3. Find the mixed distribution f_X of X. if the c.d.f. of X is
$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 0.2 + 0.3x & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad x \ge 2 \end{cases}$$

4. Find $F_X(2.5-x)$ for $0 < x < 0.5$ and $F_X(2.5-x)$ for $0.5 < x < 2$

5. Let X_1 and X_2 be two i.i.d. random losses having the same distribution as X. Calculate $F_{X_1+X_2}(2.5)$. (Hint fir mixed distributions $F_{X_1+X_2}(s) = \sum_k f_{X_1}(k)F_{X_2}(s-k) + \int_{-\infty}^{+\infty} f_{X_1}(x)F_{X_2}(s-x)dx$).

Solution:

1. Recall that $f_1 * f_2(x)$ is the p.m.f. of $X_1 + X_2$.

k	$f_1(k)$	$f_1 * f_2(k)$	$f_1 * f_2 * f_3(k)$	$f_1 * f_2 * f_3 * f_4(k)$
0	0.500	**0.200**	0.080	0.0240
1	0.500	0.350	**0.200**	0.0840
2	0.000	0.300	0.265	**0.1715**

2. What is
$$\mathrm{P}\left(S \geq 3\right) = 1 - \mathrm{P}\left(S \leq 2\right) = 1 - 0.024 - 0.0804 - 0.1715 = \mathbf{0.7241}.$$

3. X has a mixed distribution with probability function
$$f_X(x) = \begin{cases} \mathbf{0} & \text{if} \quad x < 0 \\ \mathbf{0.2} & \text{if} \quad x = 0 \\ \mathbf{0.3} & \text{if} \quad 0 < x < 2 \\ \mathbf{0.2} & \text{if} \quad x = 2 \\ \mathbf{0} & \text{if} \quad x > 2 \end{cases}$$

4. We have $F(2.5 - x) = \mathbf{1}$ for $0 < x < 0.5$, and $F(2.5 - x) = 0.2 + 0.3(2.5 - x) = \mathbf{0.95 - 0.3x}$ for $0.5 < x < 2)$

5. We must use a combination of the discrete and continuous approach to find the convolution
$$\begin{aligned} F_{X_1+X_2}(2.5) &= f(0) \cdot F(2.5) + \int_0^2 f(x)F(2.5 - x)\ \mathrm{d}x + f(2) \cdot F(0.5) \\ &= (0.2) \cdot (1) + \int_0^{0.5} (0.3)(1)\ \mathrm{d}x + \int_{0.5}^2 (0.3)\left(0.95 - 0.3x\right)dx + (0.2)(0.35) \\ &= \mathbf{0.67875}. \end{aligned}$$

Problem 3. The distribution of aggregate claims has a compound Poisson distribution with parameter λ. The first three values of the distribution are

k	0	1	2
$F_S(k)$	$\frac{1}{e}$	$\frac{1}{e}$	$\frac{2}{e}$

Individual claim amounts can only have **positive integral values**.

1. Calculate $f_X(0)$ and λ.
2. Find $f_S(1)$ and $f_S(2)$, (Hint $\mathrm{P}(S = k) = F_S(k) - F_S(k-1)$).
3. Find $f_X(1)$ and $f_X(2)$,
4. Deduce from 3. $f_X(x)$ for all $x \neq 2$
5. Use Panjer's recursion to find $f_S(3)$ and $f_S(4)$.
6. Calculate the probability that the aggregate loss is greater than 4.5.

Solution:

1. Since claim amounts are positive integers, $f_X(0) = \mathbf{0}$, and thus, with Poisson parameter λ,
$$f_S(0) = F_S(0) = P_N(0) = \mathrm{P}(N = 0) = e^{-\lambda} = \frac{1}{e} \text{ which implies } \lambda = \mathbf{1}.$$

2. From the given information, $f_S(1) = F_S(1) - F_S(0) = 0$, and $f_S(2) = F_S(2) - F_S(1) = \frac{1}{e} = \mathbf{0.36788}$.

3. Using Panjer's recursion when N is Poisson(1), we get

$$0 = f_S(1) = \frac{1}{1}\left[1 \cdot f_X(1) \cdot f_S(0)\right] \text{ which implies } f_X(1) = \mathbf{0},$$

and

$$\frac{1}{e} = f_S(2) = \frac{1}{2}\left[1 \cdot f_X(1) \cdot f_S(1) + 2 \cdot f_X(2) \cdot f_S(0)\right] \text{ which implies } f_X(2) = \mathbf{1}$$

4. For all $x \neq 2$ we have $f_X(x) = \mathbf{0}$ since $f_X(2) = 1$.

5. Again Panjer's recursion leads to

$$f_S(3) = \frac{1}{3}\left[1 \cdot f_X(1) \cdot f_S(2) + 2 \cdot f_X(2) \cdot f_S(1) + 3 \cdot f_X(3) \cdot f_S(0)\right] = \mathbf{0},$$

and

$$f_S(4) = \frac{1}{4}\left[2 \cdot f_X(2) \cdot f_S(2)\right] = \frac{1}{2e} = \mathbf{0.18394}.$$

6. Finally

$$\mathrm{P}(S > 4.5) = 1 - F_S(4) = 1 - \sum_{k=0}^{4} f_S(k) = 1 - \frac{1}{e} - 0 - \frac{1}{e} - 0 - \frac{1}{2e} = 1 - \frac{5}{2e} = \mathbf{0.08030}.$$

Problem 4.

The number of claims, N, made on an insurance portfolio follows the distribution:

n	0	2	3
$\mathrm{P}(N = n)$	0.7	0.2	0.1

If a claim occurs, the benefit is 0 or 10 with probability 0.8 and 0.2, respectively. The number of claims and the benefit for each claim are independent. By convention $S = 0$ if $N = 0$.

1. Calculate the expected value of the aggregate loss
2. Calculate Variance of the aggregate loss
3. Calculate the probability that aggregate benefits will exceed expected aggregate benefits by more than 2 standard deviations of the aggregate benefits.

Solution:

1. We are given the distribution of N (claims number) and X (claim amount). The mean and variance of N and X are found as follows.

$$\mathrm{E}[N] = (0)(0.7) + (2)(0.2) + (3)(0.1) = 0.7 \text{ and } \mathrm{E}[X] = (0)(0.8) + (10)(0.2) = 2,$$

so the mean of the aggregate benefit S is $\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X] = \mathbf{1.4}$.

2. Moreover

$$\mathrm{E}\left[N^2\right] = (0^2)(0.7) + (2^2)(0.2) + (3^2)(0.1) = 1.7 \text{ and } \mathrm{E}\left[X^2\right] = (0^2)(0.8) + (10^2)(0.2) = 20,$$

hence

$$\mathrm{Var}(N) = \mathrm{E}[N^2] - (\mathrm{E}[N])^2 = 1.21 \text{ and } \mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 16.$$

Thus variance of the aggregate benefits S is

$$\mathrm{Var}(S) = \mathrm{Var}(N)(\mathrm{E}[X])^2 + \mathrm{E}[N]\,\mathrm{Var}(X) = 16.04.$$

The standard deviation of S is $\mathrm{sd}(S) = \sqrt{16.04} = \mathbf{4.005}$.

3. We are asked to find

$$\mathrm{P}\left(S - \mathrm{E}\left[S\right] > 2\mathrm{sd}(S)\right) = \mathrm{P}\left(S > 9.4\right) = 1 - \mathrm{P}\left(S \leq 9.4\right) = 1 - \mathrm{P}\left(S = 0\right).$$

let us first calculate $\mathrm{P}\left(S = 0\right)$ we have

$$\begin{aligned}
\mathrm{P}\left(S = 0\right) &= \mathrm{P}\left(S = 0,\ N = 0\right) + \mathrm{P}\left(S = 0,\ N = 2\right) + \mathrm{P}\left(S = 0,\ N = 3\right) \\
&= \mathrm{P}\left(S = 0 \mid N = 0\right)\mathrm{P}\left(N = 0\right) + \mathrm{P}\left(S = 0 \mid N = 2\right)\mathrm{P}\left(\ N = 2\right) \\
&\quad + \mathrm{P}\left(S = 0 \mid N = 3\right)\mathrm{P}\left(N = 3\right) \\
&= \mathrm{P}\left(N = 0\right) + \mathrm{P}\left(X_1 = 0,\ X_2 = 0 \mid N = 2\right)\mathrm{P}\left(\ N = 2\right) \\
&\quad + \mathrm{P}\left(X_1 = 0,\ X_2 = 0,\ X_3 = 0 \mid N = 3\right)\mathrm{P}\left(N = 3\right) \\
&= 0.7 + (0.8)^2 \times 0.2 + (0.8)^3 \times 0.1 = 0.8792.
\end{aligned}$$

hence $\mathrm{P}\left(S - \mathrm{E}\left[S\right] > 2\mathrm{sd}(S)\right) = 1 - 0.8792 = \mathbf{0.1208}$.

Problem 5.

1. For a discrete probability distribution, you are given the recursion relation

$$p_k = \frac{2}{k}p_{k-1}, \quad k = 1,\ 2, \ldots$$

Determine p_4.

2. An actuary has created a **compound claims frequency model** taking non–negative integer values with the following properties:
(i) The primary distribution is the negative binomial with probability generating function

$$\mathrm{P}_N(t) = \frac{1}{(1 - 3(t-1))^2}$$

(ii) The secondary distribution is the Poisson with probability generating function $\mathrm{P}_X(t) = e^{\lambda(t-1)}$
(iii) The probability of no claims equals 0.067. Calculate λ.

3. Let N be a discrete random variable with a probability function which is a member of the $(a, b, 0)$ class of distributions that describes the number of claims of an insurance company. You are given:
(i) $\mathrm{P}(N = 0) = \mathrm{P}(N = 1) = 0.25$,
(ii) $\mathrm{P}(N = 2) = 0.1875$.
Calculate the probability that the number of claims equals to 3.

Solution:

1. The Poisson distribution with parameter λ has

$$\frac{p_k}{p_{k-1}} = \frac{(k-1)!}{e^{-\lambda}\lambda^{k-1}} \frac{e^{-\lambda}\lambda^k}{k!} = \frac{\lambda}{k},$$

so that $a = 0,\ b = \lambda$. Therefore, the distribution in this problem is Poisson with $\lambda = 2$, and

$$p_4 = e^{-2}\,\frac{2^4}{4!} = e^{-2}\,\frac{2}{3} = \mathbf{0.090224}.$$

2. If S is a discrete non–negative integer random variable with probability generating function $\mathrm{P}_S(t)$, then $\mathrm{P}\left(S = 0\right) = \mathrm{P}_S(0)$. Suppose that N is the primary distribution (negative binomial) and X is the secondary (Poisson), and let S denote the compound claims frequency model. Then the probability generating function of S is

$$\mathrm{P}_S(t) = \mathrm{P}_N\left(\mathrm{P}_X(t)\right) = \left[1 - 3\left(e^{\lambda(t-1)} - 1\right)\right]^{-2}$$

Then, $0.067 = \mathrm{P}_S(0) = \left[1 - 3\left(e^{\lambda(0-1)} - 1\right)\right]^{-2}$. Solving this equation for λ results in $\lambda = \mathbf{3.1}$.

3. As a member of the $(a, b, 0)$ class of distributions, the probability mass function $p_k = \mathrm{P}(N = k)$ must satisfy $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$ for $k = 1,\ 2,\ \ldots$ From the given probabilities, we have

$$\frac{p_1}{p_0} = \frac{0.25}{0.25} = 1 = a + b,$$

and

$$\frac{p_2}{p_1} = \frac{0.1875}{0.25} = 0.75 = a + \frac{b}{2}.$$

Solving the two equations for a and b results in $a = 0.5$ and $b = 0.5$. Then

$$\frac{p_3}{p_2} = \frac{p_3}{0.1875} = a + \frac{b}{3} = 0.5 + \frac{0.5}{3}$$

which implies that $p_3 = \mathrm{P}(N = 3) = \mathbf{0.125}$.