King Saud University
College of Sciences
Mathematics Department

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Solution of the first midterm exam ACTU–464-474 (25%)

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Problem 1.

Let X denotes a loss of reinsurance company such that the c.d.f. F of this loss is given as follows

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 20 \\ \dfrac{x+20}{80} & \text{for} \quad 20 \leq x < 40 \\ 1 & \text{for} \quad x \geq 40 \end{cases}$$

1. Calculate the probability the loss is less or equal to 30
2. Calculate the probability the loss equal exactly 40
3. Calculate the expected loss and the standard deviation of this loss.

Solution:

1. Notice first that the p.m.d.f. of X is given by

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 20 \\ \dfrac{1}{2} & \text{for} \quad x = 20 \\ \dfrac{1}{80} & \text{for} \quad 20 < x < 40 \\ \dfrac{1}{4} & \text{for} \quad x = 40 \\ 0 & \text{for} \quad x > 40 \end{cases}$$

$\mathrm{P}(X \leq 30) = F(30) = \dfrac{30+20}{80} = \frac{5}{8} = \mathbf{0.625}$.

2. $\mathrm{P}(X = 40) = \mathbf{\dfrac{1}{4}}$.

3. The expected loss is

$$\begin{aligned} \mathrm{E}[X] &= 20\mathrm{P}(X=20) + \int_{20}^{40} x \frac{1}{80} dx + 40\mathrm{P}(X=40) = \frac{20}{2} + \frac{1}{160}\left[x^2\right]_{20}^{40} + \frac{40}{4} \\ &= 10 + \frac{1}{160}\left(40^2 - 20^2\right) + \frac{40}{4} = \frac{55}{2} = 27.5. \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\left[X^2\right] &= 20^2\mathrm{P}(X=20) \int_{20}^{40} x^2 \frac{1}{80} dx + 40^2\mathrm{P}(X=40) = \frac{20^2}{2} + \frac{1}{240}\left[x^3\right]_{20}^{40} + \frac{40^2}{4} \\ &= \frac{20^2}{2} + \frac{1}{240}\left(40^3 - 20^3\right) + \frac{40^2}{4} = \frac{2500}{3} = 833.33 \end{aligned}$$

and the $\mathrm{sd}(X) = \sqrt{833.33 - (27.5)^2} = \mathbf{8.7795}$.

Problem 2.

Consider a loss X having the following distribution

$$f_X(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ \frac{1}{2} & \text{if} \quad x = 1 \\ x-1 & \text{if} \quad 1 < x < 2 \\ 0 & \text{if} \quad x \geq 2. \end{cases}$$

1. Calculate the safety loading premium $\Pi_{\text{SL}}(0.05)$ and $\text{P}(X \leq \Pi_{\text{SL}}(0.05))$
2. Calculate the σ–loading premium $\Pi_{\text{sd}}(0.05)$ and $\text{P}\left(X > \Pi_{\text{sd}}(0.05)\right)$
3. Calculate the exponential premium $\Pi_{\text{Exp}}(0.05)$ and $\text{P}(X \leq \Pi_{\text{Exp}}(0.05))$.

Solution:

1. We know that $\Pi_{\text{SL}}(0.05) = 1.05\text{E}\left[X\right]$. Thus

$$\Pi_{\text{SL}}(0.05) = 1.05\text{E}\left[X\right] = 1.05(0.5 + \int_1^2 x\left(x-1\right)dx) = \left(1 + \frac{5}{100}\right) \times \frac{4}{3} = \frac{7}{5} = \mathbf{1.4}.$$

And

$$\text{P}(X \leq \Pi_{\text{SL}}(0.05)) = \text{P}(X \leq 1.4) = \frac{(1.4)^2 - 2 \times 1.4 + 2}{2} = \mathbf{0.58}.$$

2. By definition $\Pi_{\text{sd}}(0.05) = \text{E}\left[X\right] + 0.05\sqrt{\text{Var}(X)}$, $\text{E}\left[X^2\right] = 0.5 + \int_1^2 x^2\left(x-1\right)dx) = 1.9167$ and $\text{Var}(X) = 1.9167 - (1.3333)^2 = 0.13901$. Hence

$$\Pi_{\text{sd}}(0.05) = 1.3333 + 0.05\sqrt{0.13901} = \mathbf{1.3519}.$$

Observe first that

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ \frac{1}{2} + \frac{(x-1)^2}{2} & \text{if} \quad 1 \leq x < 2 \\ 1 & \text{if} \quad x \geq 2, \end{cases}$$

therefore

$$\begin{aligned} \text{P}\left(X \leq \Pi_{\text{sd}}(0.05)\right) &= \text{P}(X \leq 1.3519) \\ &= \frac{1}{2} + \frac{(1.3519-1)^2}{2} = 0.56192. \end{aligned}$$

Finally $\text{P}\left(X > \Pi_{\text{sd}}(0.05)\right) = 1 - 0.56192 = \mathbf{0.43808}$.

3. By definition $\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln\left(M_X(0.05)\right)$, and

$$M_X(0.05) = \frac{1}{2}e^{0.05} + \int_1^2 e^{0.05x}\left(x-1\right)dx = 1.0691,$$

therefore

$$\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln\left(1.0691\right) = \mathbf{1.3363}.$$

And

$$\begin{aligned} \text{P}(X &\leq \Pi_{\text{Exp}}(0.05)) = \text{P}(X \leq 1.3363) \\ &= \frac{1}{2} + \frac{(1.3363-1)^2}{2} = \mathbf{0.55655} \end{aligned}$$

Problem 3.

A random loss X has m.g.f. $M_X(t) = (1-2t)^{-5}$ for $t < 0.5$.

1. Calculate the exponential premium $\Pi_{\text{Exp}}(0.398\,41)$ of the loss X.
2. Use normal approximation to calculate $\Pi_{0.05}$ and $\Pi_{0.01}$ such that $\text{P}(X > \Pi_\alpha) = \alpha$. What is the interpretation of Π_α when X is a loss of an insurance company?
3. Consider a risk whose distribution follows a Pareto distribution with parameters $\alpha = 2.5$ and $\theta = 50$. The c.d.f. of a Pareto distribution with parameters α and θ is

$$F_{\text{Pareto}}(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha, \quad \text{E}[X] = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \text{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}.$$

Calculate the parameter b of the σ–loading premium $\Pi_{\text{sd}}(b)$ such that $\text{P}(X \geq \Pi_{\text{sd}}(b)) = 0.08$.

Solution:

1. We have $\Pi_{\text{Exp}}(0.398\,41) = \frac{1}{0.398\,41} \ln\left((1 - 2 \times 0.398\,41)^{-5}\right) = \mathbf{20}$.
2. First we calculate $\text{E}[X] = M'_X(0) = 10$ and $\text{sd}(X) = \sigma_X = \sqrt{M''_X(0) - (M'_X(0))^2} = 2\sqrt{5}$

$$\text{P}(X > \Pi_\alpha) = \text{P}\left(\frac{X - 10}{2\sqrt{5}} > \frac{\Pi_\alpha - 10}{2\sqrt{5}}\right) = \text{P}\left(Z > \frac{\Pi_\alpha - 10}{2\sqrt{5}}\right) = \alpha,$$

so $\frac{\Pi_{0.05} - 10}{2\sqrt{5}} = 1.644854$, thus

$$\Pi_{0.05} = 10 + 1.644854 \times 2\sqrt{5} = \mathbf{17.356},$$

and $\frac{\Pi_{0.01} - 10}{2\sqrt{5}} = 2.326348$, then

$$\Pi_{0.01} = 10 + 2.326348 \times 2\sqrt{5} = \mathbf{20.404}.$$

If X is a loss of an insurance company then Π_α reprsents the precentile premium associated to the loss X with the risk α?

3. The premium $\Pi_{\text{sd}}(b)$ is given by

$$\left(\frac{50}{50 + \Pi_{\text{sd}}(b)}\right)^{2.5} = 0.08 \quad \text{that is} \quad \Pi_{\text{sd}}(b) = 87.32,$$

Remember that

$$\Pi_{\text{sd}}(b) = \text{E}[X] + b\sqrt{\text{Var}(X)} = \frac{\theta}{\alpha - 1}\left(1 + b\sqrt{\frac{\alpha}{(\alpha - 2)}}\right) = \frac{50}{1.5}\left(1 + b\sqrt{5}\right) = 87.32,$$

hence $b = \mathbf{0.72431}$.

Problem 4.

An insurer undertakes a risk X distributed as follows $\text{P}(X = 0) = 1 - \text{P}(X = 36) = \dfrac{2}{3}$ and after collecting the premium, he owns a capital $W = 100$.

1. What is the maximum premium P^+ the insurer is willing to pay to a reinsurer to take over the complete risk, if his utility function is $u(x) = \ln(x)$ for $x > 0$?
2. Calculate the net premium denoted by μ of the risk X and its variance σ^2.
3. Find the approximation P_a^+ of P^+ where $P_a^+ = \mu - \dfrac{\sigma^2}{2}\dfrac{u''(W - \mu)}{u'(W - \mu)}$ and compare P_a^+ and P^+.

Solution:

1. The the maximum amount P^+ is given by the equation

$$\begin{aligned} u(W-P^+) &= \mathrm{E}\left[u\left(W-X\right)\right] \iff u\left(100-P^+\right) = \frac{2u\left(100\right)}{3} + \frac{1u\left(64\right)}{3} \\ &\iff 3\ln\left(100-P^+\right) = 2\ln(100) + 2\ln\left(64\right) = \ln\left(\left(100^2\right)\left(64\right)\right) \end{aligned}$$

   hence $\left(100-P^+\right)^3 = (100)^2\,(64) = 640000$, then $P^+ = 100 - (640000)^{\frac{1}{3}} = \mathbf{13.823}$.

2. Observe first that $\mu = \frac{1}{3}36 = 12$, $\sigma^2 = \frac{1}{3}\left(36\right)^2 - (12)^2 = \mathbf{288}$.

3. We have $\frac{u''(x)}{u'(x)} = -\frac{1}{x}$, therefore

$$P_a^+ = 12 + \frac{288}{2}\frac{1}{100-12} = \mathbf{13.636}.$$

   We observe that $P_a^+ < P^+$.

Problem 5.

A portfolio of independent insurance policies has three classes of policies:

Class	Number in Class	Probability of Claim per Policy	Claim Amount b_k
1	1000	0.01	2
2	2000	0.02	2
3	500	0.04	3

1. Calculate the expectation and variance of the aggregate loss S
2. Use normal approximation to calculate θ such that the probability of that the aggregate loss is less than the $\Pi_{\mathrm{SL}}(\theta)$ is equal to 0.95.
3. Find $\Pi_{\mathrm{SL}}(\theta)$.

Solution:

1. We have $\mathrm{E}\left[S\right] = \sum_{i=1}^{3} n_k\ b_k\ q_k = 1000 \times 2 \times 0.01 + 2000 \times 2 \times 0.02 + 500 \times 3 \times 0.04 = \mathbf{160}$. And

$$\begin{aligned} \sigma_S^2 &= \mathrm{Var}\left(S\right) = \sum_{i=1}^{3} n_k\ b_k^2\ q_k\left(1-q_k\right) \\ &= 1000 \times 2^2 \times 0.01 \times 0.99 + 2000 \times 2^2 \times 0.02 \times 0.98 \\ &\quad +500 \times 3^2 \times 0.04 \times 0.96 \\ &= \mathbf{369.20} \end{aligned}$$

2. Under normal approximation the r.v. $T = \frac{S-\mathrm{E}[S]}{\sigma_S}$ follows a standard normal distribution, therefore

$$\mathrm{P}\left(S \le \Pi_{\mathrm{SL}}(\theta)\right) = \mathrm{P}\left(\frac{S-\mathrm{E}\left[S\right]}{\sigma_S} \le \frac{\Pi_{\mathrm{SL}}(\theta)-160}{\sqrt{369.20}} = \theta\frac{160}{\sqrt{369.20}}\right) = 0.95,$$

   hence $\theta = \frac{1.644854 \times \sqrt{369.20}}{160} = \mathbf{0.19753}$.

3. The safety loading premium is $\Pi_{\mathrm{SL}}(0.19753) = 1.19753 \times 160 = \mathbf{191.60}$.