King Saud University
College of Sciences
Mathematics Department

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Solution of the final exam ACTU-464-474 Spring 2021 (20%)
April 19, 2021 (three hours: from 1:30 to 4:30 PM)
Keep four digits after dot and round by default there after.

Exercise 1 ***(4 marks)***

1. *An actuary have shown from available data that the number of claims in an insurance company is a member of $C(a,b,0)$ class of distributions such that $p_0 = p_1 = \frac{1}{4}$, $p_2 = \frac{3}{16}$. Find the expected number of claims.*
2. *Suppose that the number of claims N has a Poisson distribution with mean $\lambda = 3$. Let $(p_n)_{n\geq 0}$ denotes the probability mass function of N. Calculate $\frac{p_{1442}}{p_{1441}}$.*

Solution:

1. We know that

$$p_n = p_{n-1}\left(a + \frac{b}{n}\right) \text{ for all } n \geq 1$$

   (a) We have $p_1 = p_0(a+b)$ and $p_2 = p_1(a+b)$

$$a + b = 1 \text{ and } a + \frac{b}{2} = \frac{3}{4}$$

   this gives $a = \frac{1}{2}, b = \frac{1}{2}$,

   (b) N is Negative binomial $\mathcal{NB}(r;p)$ where $p = 1 - a = \frac{1}{2}$ and $r = 2$

$$p_4 = p_3\left(a + \frac{b}{4}\right) = p_2\left(a + \frac{b}{3}\right)\left(a + \frac{b}{4}\right) = \frac{3}{16}\frac{1}{4}\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right) = \frac{5}{64} = 0.08125.$$

   (c) $\mathrm{E}[N] = \frac{rq}{p} = 2$. (since $p = q$).

2. We know that $p_n = e^{-\lambda}\frac{\lambda^n}{n!}$ thus

$$\frac{p_{1442}}{p_{1441}} = e^{-3}\frac{3^{1442}}{1442!}\frac{1441!}{3^{1441}}e^{3} = \frac{3}{1442} = 0.00208.$$

Exercise 2 ***(4 marks)*** *Losses in* 2020 *follow the density function* $f_X(x) = 3x^{-4}$, $x > 1$ *and* 0 *otherwise where* X *is the loss in millions of dollars. Inflation of* 5% *impacts all claims uniformly from* 2020 *to* 2021.

1. *Determine the c.d.f. of losses for* 2021
2. *Find the probability that a* 2021 *loss exceeds* 2.1 *millions of dollars.*

Solution:

1. The losses in 2021 can be written as $Y = 1.05X$ hence for $x > 1.1$,

$$F_Y(x) = \mathrm{P}\left(Y \le x\right) = \mathrm{P}\left(X \le \frac{x}{1.05}\right) = F_X\left(\frac{x}{1.05}\right) = \int_1^{\frac{x}{1.05}} 3t^{-4}dt = \left[-\frac{1}{t^3}\right]_1^{\frac{x}{1.05}} = 1 - \frac{(1.05)^3}{x^3}.$$

2. The required probability is

$$\mathrm{P}\left(Y > 2.1\right) = 1 - F_Y(2.1) = \frac{(1.05)^3}{(2.1)^3} = \frac{1}{2^3} = 0.125.$$

Exercise 3 ***(4 marks)*** *A portfolio of independent insurance policies has three classes of policies:*

Class	*Number in Class*	*Probability of Claim per Policy*	*Claim Amount* b_k
1	1000	0.1	30
2	2000	0.2	20
3	500	0.3	10

1. *Calculate the expectation and variance of the aggregate loss* S.
2. *Use normal approximation to calculate* Π_{SL}*such that the probability of that the aggregate loss is less or equal than the* Π_{SL} *is equal to* 0.95.

Solution:

1. We have $\mathrm{E}\left[S\right] = \sum_{i=1}^{3} n_k\ b_k\ q_k = 1000 \times 30 \times 0.1 + 2000 \times 20 \times 0.2 + 500 \times 10 \times 0.3 = \mathbf{12500}$. And

$$\begin{aligned}\sigma_S^2 &= \mathrm{Var}\left(S\right) = \sum_{i=1}^{3} n_k\ b_k^2\ q_k\left(1 - q_k\right) \\ &= 1000 \times 30^2 \times 0.1 \times 0.9 + 2000 \times 20^2 \times 0.2 \times 0.8 + 500 \times 10^2 \times 0.3 \times 0.7 = \mathbf{2195 \times 10^5}.\end{aligned}$$

2. Under normal approximation the r.v. $T = \frac{S - \mathrm{E}[S]}{\sigma_S}$ follows a standard normal distribution, therefore

$$\mathrm{P}\left(S \le \Pi_{\mathrm{SL}}\right) = \mathrm{P}\left(\frac{S - \mathrm{E}\left[S\right]}{\sigma_S} \le \frac{\Pi_{\mathrm{SL}} - 12500}{1000\sqrt{2195}}\right) = 0.95,$$

The safety loading premium is $\Pi_{\mathrm{SL}} = 12500 + 1.644845 \times 1000\sqrt{2195} = \mathbf{89562}$.

Exercise 4 ***(4 marks)*** *For an insured portfolio, you are given:*
(i) the number of claims has a geometric distribution with $\beta = \frac{1}{3}$*,*
(ii) individual claim amounts can take on values 3, 4 *or* 5*, with equal probability,*
(iii) the number of claims and claim amounts are independent, and
(iv) the premium charged equals expected aggregate claims plus the variance of aggregate claims.
The p.m.f. of a geometric distribution with β *is given by* $p_n = \mathrm{P}(N = n) = \frac{\beta^n}{(1+\beta)^{n+1}} = q^n p$ *for* $n \geq 0$ *with* $p = 1 - \frac{\beta}{1+\beta} = \frac{1}{1+\beta}$.

1. *Calculate the expected value and the variance of the aggregate claims.*
2. *Determine the exact probability that aggregate claims exceeds the premium given that* $F_S(8) = \frac{31}{32}$.

Solution:

1. We know that $\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}[X] = \frac{1}{3} \times 4 = \mathbf{\frac{4}{3}} = \mathbf{1.3333}$, and

$$\mathrm{Var}(S) = \mathrm{E}[N]\,\mathrm{Var}(X) + \mathrm{Var}(N)\,(\mathrm{E}[X])^2 = \frac{1}{3} \times \frac{2}{3} + \frac{4}{9} \times (4)^2 = \mathbf{\frac{22}{3}} = \mathbf{7.3333}.$$

2. The Premium is $= \frac{4}{3} + \frac{22}{3} = \frac{26}{3} = 8.6667$. Since S is integer–valued we have

$$\mathrm{P}\left(S > \frac{26}{3}\right) = \mathrm{P}(S > 8) = 1 - \mathrm{P}(S \leq 8) = 1 - F_S(8) = 1 - \frac{31}{32} = \frac{1}{32} = \mathbf{0.03125}.$$

Exercise 5 ***(4 marks)***

1. *Find the **quantile premium** with a risk of* 5% *for loss random variable* X *which is exponentially distributed with mean* $\theta = 667.6236$.
2. *The c.d.f. of aggregate losses* S *covered under a policy of stop–loss insurance is given by* $F_S(x) = 1 - \frac{4}{x^2}$, $x > 2$. *Calculate* $\mathrm{E}[1000\max(S-10;0)]$. *(Hint: If* X *is r.v. with c.d.f.* F_X *then* $\mathrm{E}[\max(X-K,0)] = \int_K^\infty (1 - F_x(x))\,dx$*).*

Solution:

1. The equation $S_X(x) = e^{-\frac{x}{\theta}} = 0.05$ implies that $x = -\theta\ln(0.05) = 2.9957\theta$ hence $\Pi_{0.05} = 2.9957 \times 667.6236 = \mathbf{2000}$.
2. We have

$$\mathrm{E}\left[(S-10)^+\right] = \int_{10}^\infty (x-10)\, f_S(x)dx = \int_{10}^\infty (x-10)\,\frac{8}{x^3}dx = \frac{2}{5}$$

thus

$$\mathrm{E}\left[1000(S-10)^+\right] = \frac{2000}{5} = \mathbf{400}.$$

Alternatively we can use

$$\mathrm{E}\left[1000(S-10)^+\right] = 1000\int_{10}^\infty (1 - F_S(x))\,dx = 1000\int_{10}^\infty \frac{4}{x^2}dx = \frac{2000}{5} = \mathbf{400}.$$