Financial Mathematics

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Contents covered during the course in Chapter One: **Time Value of Money**

- Lectures: class notes and slides
- Section 1.12 (ACTEX) examples exercises
- some extra exercise from Actex: Sample Exam PE6-1: Q2.
- Qs from 2018 sample Exam: 1,3, 9, 10

Main Content

Basic Definitions.

Principal. The initial of amount of money (capital) invested.

Time. The time from date of investment.

Period. The unit in which time is measured (days, months, years, decades, etc).

Note. The most common measurement is one year and this will be assumed unless stated otherwise.

Amount of interest (Interest). Amount of interest is the difference between the accumulated value (final balance) and the principal (invested amount):

$$I = \text{final balance} - \text{invested amount}$$

Effective rate of interest. is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period:

$$i = \frac{\text{final balance} - \text{invested amount}}{\text{invested amount}}$$

Notes.

- The effective rate of interest is expressed in percentage, for example, $i = 8\%$.
- If the invested amount (the principal) is k, then

$$i = \frac{I}{k} \Rightarrow I = i\ k$$

Section 1.1: Time Value of Money

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Example. A man invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.
(1) Find the amount of interest earned by the man in those 6 months.

(2) Find the (semi–annual) effective rate of interest earned in those 6 months.

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Solution.
(1) The amount of interest earned by Simon in those 6 months is $I = 1049.23 - 1000 = 49.23$.

(2) The (semi–annual) effective rate of interest earned is

$$i = \frac{1049.23 - 1000}{1000} = 0.004923 = 0.4923\%$$

Section 1.1: Time Value of Money

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- **Accumulated value.** The total amount received after a period of time.
- **Accumulation function $a(t)$.** gives the accumulated value at time $t \geq 0$ of an original investment of 1 where $a(0) = 1$.
- **Amount function $A(t)$.** gives the accumulated value at time $t \geq 0$ of an original investment of k:

$$A(t) = A(0)\ .\ a(t) \quad \text{for} \quad A(0) = k, \quad \text{we have} \quad A(t) = k\ .\ a(t)$$

Notes.

- The accumulation function $a(t)$ is a special case of the amount function $A(t)$ for which $k = 1$.
- For each $t \geq 0$, $A(t) > 0$.
- The amount function $A(t)$ is non–decreasing.

- Suppose that an amount $A(0)$ of money is invested at time 0. Then,
- $A(0)$ is the principal.
- $A(t)$ is the value at time t of the initial investment $A(0)$.
- The amount of interest earned over the period $[s, t]$: interest earned at t minus interest earned at s

$$I_t = I_{[s,t]} = A(t) - A(s)$$

- The effective rate of interest earned in the period $[s, t]$:

$$i_t = i_{[s,t]} = \frac{A(t) - A(s)}{A(s)} = \frac{a(t) - a(s)}{a(s)}$$

Special Case 1. The effective rate of interest earned in the first period $[0, 1]$:

$$i_1 = i_{[0,1]} = \frac{A(1) - A(0)}{A(0)} = \frac{I}{A(0)} \Rightarrow I = ik,$$

where $A(0)$ is the amount of money invested at time zero (principal).

Special Case 2. The effective rate of interest earned in the nth period $[n-1, n]$:

$$i_n = I_{[n-1,n]} = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}, \quad \text{for integer } n \geq 1$$

Note. The effective rate over a 1–year period of time is called **the annual effective rate**.

Example. A man invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)(1 + \frac{t}{40})$, where t is the number of years after March 1, 2008.

(1) Find the balance in the man account on October 1, 2008.

(2) Find the amount of interest earned in those 7 months.

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$$\frac{A(7/12) - A(0)}{A(0)} = \frac{72.917}{5000} = 0.0145834 = 1.45834\%$$

Rule 1. **Proportionality**

If an investment strategy follows the amount function $A(t)$, $t > 0$, an investment of \$k made at time 0 with the previous investment strategy, has a value of \$ $k\frac{A(t)}{A(0)}$ at time t.

Recall. The accumulation function $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.

By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.

Knowing the value function $a(t)$ and the principal $A(0)$, we can find the amount function $A(t)$ using the formula $A(t) = A(0)a(t)$.

$$\text{This implies } A(t) = A(0)a(t) \Rightarrow a(t) = \frac{A(t)}{A(0)} \Rightarrow k.a(t) = k.\frac{A(t)}{A(0)}$$

Notes.

Investing $A(0)$ at time zero, we get $A(t)$ at time t.

Investing 1 at time zero, we get $\frac{A(t)}{A(0)}$ at time t.

Section 1.2: Present Value and Future Value

Present value (PV). The present value at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.

Note. Let x be the amount which need to invest at time zero to get a balance of k at time t. So, we have

$$k = x\frac{A(t)}{A(0)} \Rightarrow x = k\frac{A(0)}{A(t)} = PV \ .$$

Hence, the present value at time 0 of a balance of k had at time t is $k\frac{A(0)}{A(t)}$.

Accumulated value or future value (FV). The accumulated value at certain time in the future of a cashflow is the amount obtained by investing at time 0.

Note. If we invest k at time zero, we get at time t:

$$FV = k\frac{A(t)}{A(0)}$$

We say that: the accumulated value at time t (sometimes we say: the present value at time t) of a deposit of k made at time zero is \$ $k\frac{A(t)}{A(0)}$.

Summary. Using the accumulation function $a(t)$, $t \geq 0$, we have:

Present Value	**Future Value**
The present value at time 0 of a balance of k had at time t: $\frac{k}{a(t)} = k\frac{A(0)}{A(t)}$	The accumulated value at time t (the balance or the present value at time t) of a deposit of k made at time zero: $ka(t) = k\frac{A(t)}{A(0)}$

Example. The accumulation function of a fund is $a(t) = (1.03)^{2t}, \ t \geq 0$.
(1) A man invests \$5000 at time zero in this fund. Find the balance into the man fund at time 2.5 years.

(2) How much money does the man need to invest into the fund at time 0 to accumulate \$10000 at time 3?

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(2) The amount which the man needs to invest at time 0 to accumulate \$10000 at time 3 is

$$PV = \frac{k}{a(t)} = \frac{10000}{a(3)} = \frac{10000}{(1.03)^{2(3)}} = 8374.842567$$

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- **Cashflow:** a series of payments (deposits/withdrawals) made at different times.

The payments can be either made by the individual or to the individual.

- An inflow is payment to the individual and represented by positive numbers.
- An outflow is a payment by the individual and represented by negative numbers.

Consider a situation in which an investor makes deposits or contributions into an investment of $C_0, C_1, \ldots, C_n$ at times $t_1, t_2, t_3, \ldots, t_n$. Then,

- If $C_t > 0$, there is a net cash flow into the investment at time t.
- If $C_t < 0$, there is a net cash flow out the investment at time t

If deposits/withdrawals are made according with the table

Time (in periods)	t_1	t_2	...	t_n
Investments	C_1	C_2	...	C_n

Rule 2. **Grows-depends-on-balance rule**

If an investment follows the amount function $A(t)$, $t \geq 0$, the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.

If an account has a balance of k at time t and no deposits/withdrawals are made in the future, then the future balance in this account does not depend on how the balance of k at time t was attained.

Further Clarification. The following two accounts have the same balance for times bigger than s:
1. An account where a unique deposit of k is made at time s.
2. An account where a unique deposit of $\frac{k}{A(s)}$ is made at time zero.

Time	0	s	t
Contribution		k	$k\frac{A(t)}{A(s)}$
	$\frac{k}{A(s)}$	k	$k\frac{A(t)}{A(s)}$

Theorem. For $t > s$, if an investment follows the amount function $A(t)$,

(1) The present value at time t (the future value at t) of a deposit of \$k made at time s is \$ $k\frac{A(t)}{A(s)} = k\frac{a(t)}{a(s)}$.

(2) The present value at time s of a balance of k had at time t is $k\frac{A(s)}{A(t)} = k\frac{a(s)}{a(t)}$.

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Example. The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$.
(1) Ali invests \$3500 into the fund at time 1. Find the value of Ali's fund account at time 4.

(2) How much money needs Sara to invest at time 2 to accumulate \$700 at time 4.

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Example. The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$.
(1) Ali invests \$3500 into the fund at time 1. Find the value of Ali's fund account at time 4.

(2) How much money needs Sara to invest at time 2 to accumulate \$700 at time 4.

Solution.

(1) The value of Alis account at time 4 is $3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000$

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Solution.

(1) The value of Alis account at time 4 is $3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000$

(2) To accumulate \$700 at time 4, Sara needs to invest at time 2, $700\frac{a(2)}{a(4)} = 700\frac{1+\frac{2}{20}}{1+\frac{4}{20}} = 700\frac{1.1}{1.2} = 641.67$

Section 1.2: Present Value and Future Value

Theorem. Present value of a cashflow. If an investment account follows the amount function $A(t),\ t > 0$, and $0 \le t_1 < t_2 < \cdots < t_n$

Time	t_1	t_2	...	t_n
Deposits	C_1	C_2	...	C_n

- The future value at time t of the cashflow: $FV(t) = \sum_{j=1}^{n} C_j \frac{A(t)}{A(t_j)} = \sum_{j=1}^{n} C_j \frac{a(t)}{a(t_j)}$

- The present value at time zero: $PV(t) = \sum_{j=1}^{n} C_j \frac{1}{a(t_j)}$

Proof.

Time	t_1	t_2	t_3	...	t_n
Balance before deposit	0	$C_1 \frac{a(t_2)}{a(t_1)} = \sum_{j=1}^{1} C_j \frac{a(t_2)}{a(t_j)}$	$\frac{a(t_3)}{a(t_2)} \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)} = \sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)}$	...	$\frac{a(t_n)}{a(t_{n-1})} \sum_{j=1}^{n-1} C_j \frac{a(t_{n-1})}{a(t_j)} = \sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)}$
Balance after deposit	C_1	$\sum_{j=1}^{1} C_j \frac{a(t_2)}{a(t_j)} + \underbrace{C_2}_{C_2 \frac{a(t_2)}{a(t_2)}} = \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)}$	$\sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)} + \underbrace{C_3}_{C_3 \frac{a(t_3)}{a(t_3)}} = \sum_{j=1}^{3} C_j \frac{a(t_3)}{a(t_j)}$	...	$\sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)} + \underbrace{C_n}_{C_n \frac{a(t_n)}{a(t_n)}} = \sum_{j=1}^{n} C_j \frac{a(t_n)}{a(t_j)}$

Example. The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20},\ t > 0$. Nadia invests \$1000 into the fund at time 1 and she withdraws \$500 at time 3. Find the value of Nadia's fund account at time 5.

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Solution. The cashflow is

Time	1	3
Deposits	1000	−500

The value of Nadias account at time 5 is

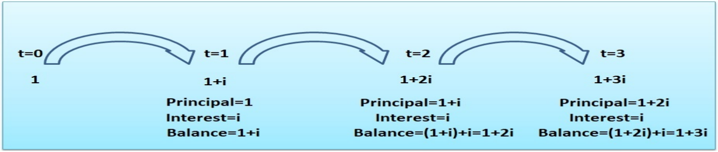
Remember. Present value $= k\frac{A(t)}{A(s)} = k\frac{a(t)}{a(s)}$

$$1000\frac{a(5)}{a(1)} - 500\frac{a(5)}{a(3)} = 1000\frac{1+\frac{5}{20}}{1+\frac{1}{20}} - 500\frac{1+\frac{5}{20}}{1+\frac{3}{20}} = 1000\frac{1.20}{1.05} - 500\frac{1.20}{1.15} = 1190.48 - 543.48 = 647.00$$

Section 1.3: Simple and Compound Interest

■ **Simple interest (S.I)** is the investment interest earned during each period so that the interest is constant.

If i is the effective annual rate of simple interest, from investing \$1, we have



Remember. $i = \frac{I}{k} \Rightarrow I = ik$ and Balance at t = Principal (amount at $(t-1)$ + Interest (I).

■ If i is the effective annual rate of simple interest, the accumulation function (balance) at time t years:

$$a(t) = (1 + it) \quad \text{for integer} \quad t \geq 0$$

■ In general, if i is the effective annual rate of simple interest, the investment of k at time zero, then

■ The amount of interest at time t years: kit.

■ The balance at time t years: $k + kit = k(1 + it)$, i.e

$$A(t) = k(1 + it) \quad \text{for integer} \quad t \geq 0$$

- If i is the effective annual rate of simple interest, the investment of k at time s, then
- The amount of interest at time t years $(t > s)$: $ki(t-s)$.

$$ka(t) - ka(s) = k(1+it) - k(1+is) = kit - kis = ki(t-s)$$

- The balance at time t years: $A(t-s) = k + ki(t-s) = k\big(1 + i(t-s)\big)$, i.e
- The effective rate of interest for $[s, t]$: $i_{[s,t]} = \frac{a(t)-a(s)}{a(s)} = \frac{ki(t-s)}{1+is}$
- Over multi–year periods a simple interest of i leads to declining effective rates over time:

$$i_{[1,2]} > i_{[2,3]} > i_{[3,4]} > \cdots > i_{[n-1,n]}$$

- The present value at time s of a balance of x at time t.
- To find the amount of deposit at time s to get a balance of \$ x in time t where $s < t$, we use the above formula as follows:

$$A(t-s) = x \Rightarrow k\ \big(1 + i(t-s)\big) = x \Rightarrow k = \frac{x}{\big(1 + i(t-s)\big)} = PV$$

- To find the amount of deposit at time zero to get a balance of \$ x in time t: $k = \frac{x}{(1+it)} = PV$

Example. A man invest \$2000 for four years if the rate of simple interest is 8% annum, find the following:

(1) The accumulation value, (2) The amount of interest.

- If i is the effective annual rate of simple interest, the investment of k at time s, then
- The amount of interest at time t years ($t > s$): $ki(t-s)$.

$$ka(t) - ka(s) = k(1+it) - k(1+is) = kit - kis = ki(t-s)$$

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Solution.

(1) The accumulation value: $A(t) = k(1+it) \Rightarrow A(4) = 2000(1+0.08(4)) = 2640$

(2) The amount of interest: $I = \text{Balance} - \text{invested amount}$: $I = 2640 - 2000 = 640$
OR use the formula: $I = kit \Rightarrow I = (2000)(8\%)(4) = 640$

Section 1.3: Simple and Compound Interest

Theorem. If deposits/withdrawals are make according with the table,

Time	t_1	t_2	t_3	...	t_n
Deposits	C_1	C_2	C_3	...	C_n

where $0 \leq t_1 < t_2 < \cdots < t_n$ to an account earning **simple interest** with annual effective rate of i , then the **balance** at time t years, where $t > t_n$, is given by

$$B = \sum_{j=1}^{n} C_j \Big(1 + i(t - t_j)\Big) = \sum_{j=1}^{n} C_j + \sum_{j=1}^{n} C_j \; i(t - t_j)$$

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Example. A man invests \$1000 into a bank account which pays simple interest with an annual rate of 7%. Nine months later, This man withdraws \$600 from the account. Find the balance in the account one year after the first deposit was made.

Solution.

Time	0	0.75
deposit/withdrawal	1000	-600

Note: nine months later: $\frac{9}{12} = \frac{3}{4} = 0.75$
The balance one year after the first deposit was made is

$$B = \sum_{j=1}^{n} C_j \Big(1 + i(t - t_j)\Big) = (1000)(1 + (1 - 0)(0.07)) + (-600)(1 + (1 - 0.75)(0.07)) = 459.5$$

Section 1.3: Simple and Compound Interest

Theorem. If deposits/withdrawals are make according with the table,

Time	t_1	t_2	t_3	$\dots$	t_n
Deposits	C_1	C_2	C_3	$\dots$	C_n

where $0 \leq t_1 < t_2 < \cdots < t_n$ to an account earning simple interest and the balance at time t years, where $t > t_n$, is B, then the annual effective rate of i

$$i = \frac{B - \sum_{j=1}^{n} C_j}{\sum_{j=1}^{n} C_j (t - t_j)}$$

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Time	t_1	t_2	t_3	...	t_n
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Example. On September 1, 2006, A man invested \$25000 into a bank account which pays simple interest. On March 1, 2007, he made a withdrawal of 5000. The accumulated value of the bank account on July 1, 2007 was \$20575. Calculate the annual effective rate of interest earned by this account.

Solution. Let September 1, 2006 be time 0. Then, March 1, 2007 is time $\frac{6}{12}$ years; and July 1, 2007 is time $\frac{10}{12}$ years.

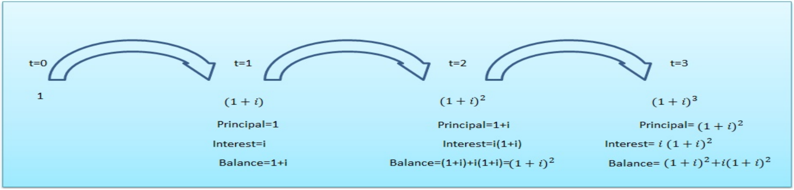
Time	Sept. 1, '06 0	Mar. 1, '07 $\frac{6}{12}$	Jul. 1, '07 $\frac{10}{12}$
deposit/withdrawal	25000	-5000	20575

The annual effective rate of interest earned by this account is

$$i = \frac{B - \sum_{j=1}^{n} C_j}{\sum_{j=1}^{n} C_j (t - t_j)} = \frac{20575 - 25000 + 5000}{25000\left(\frac{10}{12} - 0\right) - 5000\left(\frac{10}{12} - \frac{6}{12}\right)} = \frac{575}{19166.67} = 3\%$$

■ **Compound interest** is interest that is automatically reinvested. Meaning that the total investment of principal and interest earned to date is kept invested at all times.

If i is the effective annual rate of compound interest, from investing \$1, we have



Remember. $i = \frac{I}{k} \Rightarrow I = ik$ and Balance at t = Principal (amount at $(t-1)$) + Interest (I).

Note. The balance $1+i$ can be considered as principal at the beginning of the second period and will earn interest of $i(1+i)$. The accumulated value (the balance) at the end of the second period is $(1+i)+i(1+i)=(1+i)^2$. Same thing for the rest of the periods.

■ If i is the effective annual rate of compound interest, the accumulation function (balance) at time t years:

$$a(t) = (1+i)^t \quad \text{for integer} \quad t \geq 0$$

■ In general, if i is the effective annual rate of compound interest, the investment of k at time zero, the balance at time t years:

$$A(t) = k(1+i)^t \quad \text{for integer} \quad t \geq 0$$

- If i is the effective annual rate of compound interest, the investment of k at time s, then
- The amount of interest at time t years ($t > s$): .

$$ka(t) - ka(s) = k(1+i)^t - k(1+i)^s = k(1+i)^s\left((1+i)^{t-s} - 1\right)$$

- The effective rate of interest for $[s, t]$: $i_{[s,t]} = \frac{a(t)-a(s)}{a(s)} = \frac{(1+i)^t-(1+i)^s}{(1+i)^s} = (1+i)^{t-s} - 1$

Note. The effective rate of interest over a certain period of time depends only on the length of this period.

- Over multi–year periods a compound interest of i leads to a constant effective rates over time:

$$i_{[0,1]} = i_{[1,2]} = i_{[2,3]} = i_{[3,4]} = \cdots = i_{[n-1,n]}$$

Example. A man invest $2000 for four years if the rate of compound interest is 8% annum, find the following:

(1) the accumulation value, (2) the amount of interest.

Section 1.3: Simple and Compound Interest

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(1) the accumulation value, (2) the amount of interest.

Solution.
(1) The accumulation value: $A(t) = k(1+i)^t$

$$A(4) = 2000(1+0.08)^4 = 2720.98$$

(2) the amount of interest: $I = \text{Balance} - \text{invested amount}$

$$I = 2720.98 - 2000 = 720.98$$

Section 1.3: Simple and Compound Interest

Theorem. If deposits/withdrawals are make according with the table,

Time	t_1	t_2	t_3	$\dots$	t_n
Deposits	C_1	C_2	C_3	$\dots$	C_n

where $0 \leq t_1 < t_2 < \cdots < t_n$ to an account earning **compound interest** with annual effective rate of i , then

- The present value of the considered cashflow at time zero:

$$PV = \sum_{j=1}^{n} C_j \frac{1}{a(t_j)} = \sum_{j=1}^{n} C_j \frac{1}{(1+i)^{t_j}} = \sum_{j=1}^{n} C_j (1+i)^{-t_j}$$

- The future value at time t of the cashflow:

$$FV = \sum_{j=1}^{n} C_j \frac{a(t)}{a(t_j)} = \sum_{j=1}^{n} C_j \frac{(1+i)^t}{(1+i)^{t_j}} = \sum_{j=1}^{n} C_j (1+i)^{t-t_j}$$

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Solution.

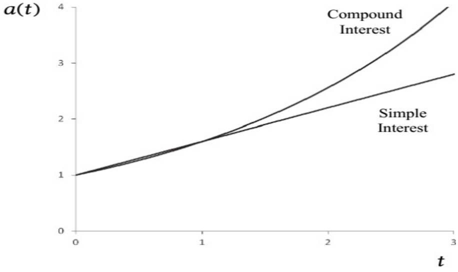
Time	0	0.75
deposit/withdrawal	1000	-600

The balance one year after the first deposit was made is

$$FV = \sum_{j=1}^{n} C_j (1+i)^{t-t_j} = 1000(1+0.07)^{(1-0)} + (-600)(1+0.07)^{(1-0.75)} = 459.765$$

- The following graph compares the growth of the two investments: simple and compound interest.

In each case, an amount of 1 is invested at time 0. One investment earns simple interest at 60% annual rate; the other earns compound interest at a 60% annual rate.



- **Notes:**

- for $t = 1$, both investments are equal (they have the same value).

- For $0 \leq t < 1$, the investment at simple interest has a larger value than the investment earning compound interest.

- For $t > 1$, the investment at compound interest has a larger value than the investment at simple interest. Also, after the first year, the compound interest grows much faster than the simple–interest investment.

Section 1.4: Present value and Discount

- **Under compound interest with effective annual rate of interest i:**

- If you invest \$$k$ for 1 year, the balance is $k(1+i)$. The quantity $(1+i)$ is called **the interest factor.**

- If you invest \$$k$ for t years, the balance is $k(1+i)^t$. The quantity $(1+i)^t$ is called **the t–year interest factor.**

- To determine how much a person must invest so that the balance will be \$1 at the end of one period:

$$1 = k(1+i) \Rightarrow k = \frac{1}{1+i} = PV$$

- The symbol $\nu = \frac{1}{1+i}$ is called **the discount factor.**

- To determine how much a person must invest so that the balance will be \$1 at the end of t years:

$$1 = k(1+i)^t \Rightarrow k = \frac{1}{(1+i)^t} = PV$$

- If you invested $\frac{1}{(1+i)^t}$ in t years ago in account earning compound interest, then the current balance is \$1.

- The quantity $\nu^t = \frac{1}{(1+i)^t}$ is called **the t–year discount factor.**

Note. For n years, we can write:

$$PV = \frac{FV}{(1+i)^n} = \frac{1}{(1+i)^n}.FV = \nu^n.FV$$

Section 1.4: Present value and Discount

Summary.

Under Simple Interest.

The quantity $a(t) = 1 + it$ is the accumulated value (the balance) of \$1 at the end of t periods. The quantity $k\ a(t) = k(1 + it)$ is the accumulated value (the balance) of \$$k$ at the end of t periods.

The quantity $a^{-1}(t) = \frac{1}{1+it}$ is the present value t period in the past (or discounted value) of \$1 to be paid at the end of t periods. The quantity $k\ a^{-1}(t) = \frac{k}{1+it}$ is the present value t periods in the past (or discounted value) of \$$k$ to be paid at the end of t periods.

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Example. Find the amount which must be invested at a rate of simple interest of 9% per annum in order to accumulate \$1000 at the end of three years.

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Solution. $k\ a^{-1}(t) = k\ \frac{1}{1+it} = 1000\ \frac{1}{1+(0.09)(3)} = \$787.40$

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Example. Find the amount which must be invested at a rate of compound interest of 9% per annum in order to accumulate \$1000 at the end of three years.

Solution. $k\ \nu^t = k\ \frac{1}{(1+i)^t} = 1000\ \frac{1}{(1+0.09)^3} = \$\ 772.18$

The effective rate of interest and Discount:

- **The effective rate of interest (i)** measures the interest earned as a percentage of the initial capital (P), and is paid or earned at the end of the period.:

$$i = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Principal}(P)} \Rightarrow i = \frac{I}{P}$$

- **The effective rate of discount (d)** measures interest as a percentage of the future value (the final balance B) and is paid at the beginning of the period.:

$$d = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Balance}(B)} \Rightarrow d = \frac{I}{B}$$

Further explanation: Take the following two cases:

Case 1. You borrow \$100 from the bank for one year at an effective rate of interest 6%. The bank will give \$100 and at the end of the year, you will repay the bank the original loan \$100 plus interest \$6, the total is \$106.

Case 2. You borrow \$ 100 from the bank for one year at an effective rate of discount 6% in advance. The bank will give \$94 and at the end of the year, you will repay the bank the original loan \$ 94 plus interest \$6, the total is \$100.

Note: In both cases, you paid \$6 interest. However, in case 1, the interest is paid at the end of the year and you had the use of \$100 for the year. In case 2, the interest is paid at the beginning of the year and you had the use of \$94 for the year.

Example. Peter invests \$738 in a bank account. One year later, his bank account is \$765.

(1) Find the effective annual interest rate earned by Peter in that year.

(2) Find the effective annual discount rate earned by Peter in that year.

The effective rate of interest and Discount:

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Example. Peter invests \$738 in a bank account. One year later, his bank account is \$765.

(1) Find the effective annual interest rate earned by Peter in that year.

(2) Find the effective annual discount rate earned by Peter in that year.

Solution. Peter earns an interest amount of $I = 765 - 738 = 27$.

(1) The effective annual interest rate earned by Peter is $i = \frac{\text{Balance}(B)-\text{Principal}(P)}{\text{Principal}(P)} = \frac{27}{738} = 3.658537\%$.

(2) The effective annual discount rate earned by Peter is $d = \frac{\text{Balance}(B)-\text{Principal}(P)}{\text{Balance}(B)} = \frac{27}{765} = 3.529412\%$.

Notes.

- The effective rate of **interest** in the period $[t-1, t]$: $i_{[t-1,t]} = \frac{a(t)-a(t-1)}{a(t-1)}$.
- The effective rate of **discount** in the period $[t-1, t]$: $d_{[t-1,t]} = \frac{a(t)-a(t-1)}{a(t)}$.

Remember.

- The quantity $(1+i)$ is called the interest factor.
- The quantity $\nu = \frac{1}{(1+i)}$ is called the discount factor.

Under the accumulation function $a(t)$, we have

- The $j-$th year interest factor is $\frac{a(j)}{a(j-1)} = \frac{(1+i)^j}{(1+i)^{j-1}} = (1+i)$.
- The $j-$th year discount factor is $\nu_j = \frac{a(j-1)}{a(j)} = \frac{(1+i)^{j-1}}{(1+i)^j} = \frac{1}{1+i}$.

Section 1.4: Present value and Discount

- Assume that a person borrows 1 at an effective rate of discount d.
- The balance $(B) = 1$
- The principal (P): $1 - d$
- The amount of interest: $I = B - P = 1 - (1-d) = \frac{1-(1-d)}{1} = \frac{B-P}{B} = d$
- From definition of the effective rate of interest:

$$i = \frac{I}{P} = \frac{d}{1-d} \Rightarrow i - di = d \Rightarrow d + di = i \Rightarrow d(1+i) = i \Rightarrow d = \frac{i}{1+i}$$

Also,

$$d = \frac{i}{1+i} = i\,\frac{1}{1+i} \Rightarrow d = i\,\nu \quad \text{and} \quad d = \frac{i}{1+i} = \frac{1+i-1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} \Rightarrow d = 1 - \nu$$

- Under compound interest,

$$d = i\nu, \quad \nu = \frac{1}{1+i}, \quad d = 1-\nu, \quad i = \frac{d}{1-d}, \quad d = \frac{i}{i+1} \quad \text{and} \quad (1-d)(1+i) = 1$$

Example. If $i = 7\%$, what are d and ν?

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Example. If $i = 7\%$, what are d and ν?

Solution. We know that $d = \frac{i}{1+i} = \frac{0.07}{1+0.07} = 6.5421\%$ and $\nu = \frac{1}{1+i} = \frac{1}{1+0.07} = 0.934579$.

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$$i = \frac{I}{P} = \frac{d}{1-d} \Rightarrow i - di = d \Rightarrow d + di = i \Rightarrow d(1+i) = i \Rightarrow d = \frac{i}{1+i}$$

Also,

$$d = \frac{i}{1+i} = i\,\frac{1}{1+i} \Rightarrow d = i\,\nu \quad \text{and} \quad d = \frac{i}{1+i} = \frac{1+i-1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} \Rightarrow d = 1 - \nu$$

- Under compound interest,

$$d = i\nu, \quad \nu = \frac{1}{1+i}, \quad d = 1 - \nu, \quad i = \frac{d}{1-d}, \quad d = \frac{i}{i+1} \quad \text{and} \quad (1-d)(1+i) = 1$$

Example. If $i = 7\%$, what are d and ν?

Solution. We know that $d = \frac{i}{1+i} = \frac{0.07}{1+0.07} = 6.5421\%$ and $\nu = \frac{1}{1+i} = \frac{1}{1+0.07} = 0.934579$.

Example. If $\nu = 0.95$, what are d and i?

Section 1.4: Present value and Discount

- Assume that a person borrows 1 at an effective rate of discount d.
- The balance $(B) = 1$
- The principal (P): $1 - d$
- The amount of interest: $I = B - P = 1 - (1-d) = \frac{1-(1-d)}{1} = \frac{B-P}{B} = d$
- From definition of the effective rate of interest:

$$i = \frac{I}{P} = \frac{d}{1-d} \Rightarrow i - di = d \Rightarrow d + di = i \Rightarrow d(1+i) = i \Rightarrow d = \frac{i}{1+i}$$

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Example. If $\nu = 0.95$, what are d and i?

Solution. We know that $d = 1 - \nu = 1 - 0.95 = 0.05$ and $\nu = \frac{1}{1+i}$, so $i = \frac{1}{\nu} - 1 = \frac{1-\nu}{\nu} = \frac{1-0.95}{0.95} = 5.2632\%$.

Section 1.4: Present value and Discount

Recall.

Under Simple Interest.

- $k.a(t) = k(1 + it)$: the accumulated value of k at the end of t periods.
- $k.a^{-1}(t) = \frac{k}{1+it}$: the present value (or discounted value) of k to be paid at the end of t periods. In terms of the amount of discount (d), we have

$$k.a^{-1}(t) = k(1 - dt) \quad \text{for} \ \ 0 \leq t \leq \frac{1}{d} \ .$$

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- $k.a(t) = (1 + i)^t$: the accumulated value of k at the end of t periods.
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$$k.a^{-1}(t) = k.\nu^t = k(1 - d)^t \quad \text{for} \ \ t \geq 0 \ .$$

Example. Find the amount which must be invested at a rate of simple discount of 9% per annum in order to accumulate \$1000 at the end of three years.

Section 1.4: Present value and Discount

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Example. Find the amount which must be invested at a rate of simple discount of 9% per annum in order to accumulate \$1000 at the end of three years.

Solution. $k \; a^{-1}(t) = k \; (1 - dt) = 1000 \; (1 - 0.09(3)) = \$730.$

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Section 1.5: Nominal rates of interest and Discount

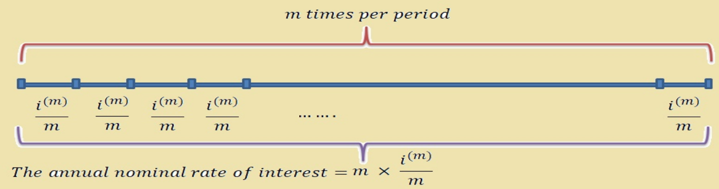
Nominal interest or discount rate. Interest is paid more frequently than once during the measurement period.

Notes.

In some cases, payments are made for a period less than a year (e.g., monthly, quartly, or semi–annually) and in this case, the period interst rate is stated as **a nominal annual rate** = the interest rate per period multiplied by the number of periods per year.

Various terms are used in practice to indicate nominal rates of interest or discount: payable, convertible and compounded.

A nominal annual interest rate is equal to the effective interest rate per period multiplied by the number of periods per year and it takes the symbol $i^{(m)}$, where $m > 1$ is a positive integer. For each m th of a period, the interest rate is $\frac{i^{(m)}}{m}$.



For example: a nominal rate of interest 8% convertible quarterly does not mean 8% per quarter but rather an interest rate of $\frac{8\%}{4} = 2$ % per quarter. If you are to earn 2% compounded quarterly, then $2\% \times 4 = 8\%$ refers to a annual nominal rate converted quarterly. So, an 8% is nominal annual rate convertible quarterly and 2% is quarterly effective rate.

Summary.

1. The interest rate per period: $\frac{i^{(m)}}{m}$.
2. The annual nominal rate: Number of periods per year × (Rate/period) $= m \times \frac{i^{(m)}}{m}$.
3. The annual effective rate: $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow i = (1 + \frac{i^{(m)}}{m})^m - 1$

Example. Suppose you are earning 2% interest compounded monthly. Find the following:
(1) The annual nominal rate?
(2) The annual effective rate?

Section 1.5: Nominal rates of interest and Discount

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Solution.
(1) The annual nominal rate ($i^{(m)}$): (Rate/period) $\times$ Number of periods per year$=$ 2% $\times$ 12 $= 0.24 = 24\%$
(2) The annual effective rate (i): $i = (1 + \frac{i^{(m)}}{m})^m - 1 \Rightarrow i = (1 + 2\%)^{12} - 1 = 0.268242 = 26.8242\%$

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Example. A man takes a loan of \$ 569. Interest in the loan is charged using compound interest. One month after a loan is taken the balance in this loan is \$ 581.

(1) Find the monthly effective interest rate, which the man is charged in his loan.

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Solution.

(1) The monthly effective interest rate, which the man is charged in his loan is

$$\frac{i^{(12)}}{12} = \frac{581 - 569}{569} = 2.108963093\%$$

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$$i^{(12)} = (12)(0.02108963093) = 25.30755712\%$$

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(3) Find the annual effective rate: $i = (1 + 2.108963093\%)^{12} - 1 = 0.28459549 = 28.459549\%$

Example. If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

Section 1.5: Nominal rates of interest and Discount

Example. If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

Solution. We solve $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + i = (1 + \frac{0.05}{4})^4 \Rightarrow i = (1 + \frac{0.05}{4})^4 - 1 \Rightarrow i = 5.0945\%$.

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$1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + 0.05 = (1 + \frac{i^{(4)}}{4})^4 \Rightarrow 1 + \frac{i^{(4)}}{4} = (1 + 0.05)^{\frac{1}{4}} \Rightarrow i^{(4)} = 4((1 + 0.05)^{\frac{1}{4}} - 1) = 4.9089\%$.

Example. If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

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- Suppose that an account follows compound interest with an annual nominal rate of interest compounded m times a year of $i^{(m)}$, then

- \$ 1 at time zero accrues to $(1+\frac{i^{(m)}}{m})$ at time $\frac{1}{m}$ years.
  - $\frac{i^{(m)}}{m}$: The ($\frac{1}{m}$-year) m–thly effective interest rate.
  - $(1+\frac{i^{(m)}}{m})$: the m-year interest factor.

- \$ 1 at time zero grows to $(1+\frac{i^{(m)}}{m})^m$ in one year.

- \$ 1 at time zero grows to $(1+\frac{i^{(m)}}{m})^{mt}$ in t years.

- The accumulation function: $a(t) = (1+\frac{i^{(m)}}{m})^{mt}$

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Example. John takes a loan of 8,000 at a nominal annual rate of interest of 10% per year convertible quarterly. How much does he owe after 30 months?

Example. If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

Solution. We solve $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + i = (1 + \frac{0.05}{4})^4 \Rightarrow i = (1 + \frac{0.05}{4})^4 - 1 \Rightarrow i = 5.0945\%$.

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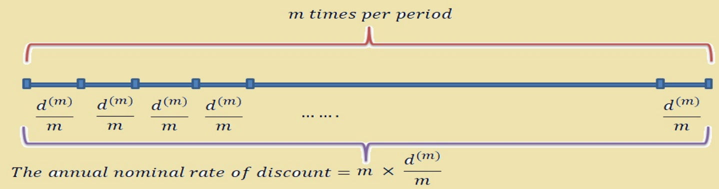
Solution. $ka(t) = k(1 + \frac{i^{(m)}}{m})^{mt} = 8000(1 + \frac{0.10}{4})^{4 \times \frac{30}{12}} = 10240.68$ Note: $\frac{30}{12} = 2.5$ i.e., two years and half.

A nominal annual discount rate is equal to the effective discount rate per period multiplied by the number of periods per year and it takes the symbol $d^{(m)}$, where $m > 1$ is a positive integer. For each m th of a period, the discount rate is $\frac{d^{(m)}}{m}$.

For example, if the quarterly effective rate of discount is 2%, we can say that the nominal annual rate of discount is 8% convertible quarterly. The annual effective rate of discount would not be 8%.

Notes.

- For each m th of a period, the discount rate is $\frac{d^{(m)}}{m}$.



- The effective rate of discount (d): $1 - d = (1 - \frac{d^{(m)}}{m})^m$

Remember. $i = \frac{d}{1-d} \Rightarrow 1 + i = 1 + \frac{d}{1-d} = \frac{1-d+d}{1-d} = \frac{1}{1-d} \Rightarrow 1 + i = (1-d)^{-1}$. From this, we have

$$1 + i = (1 + \frac{i^{(m)}}{m})^m = (1 - d)^{-1} = (1 - \frac{d^{(m)}}{m})^{-m}$$

Section 1.5: Nominal rates of interest and Discount

- Thus the accumulation function for compound interest under a nominal rate of discount $d^{(m)}$ convertible m times in t years is

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Example. If $d^{(4)} = 5\%$ find i.

Solution.

$$\begin{aligned} 1 + i = (1 - \frac{d^{(4)}}{4})^{-4} &\Rightarrow 1 + i = (1 - \frac{0.05}{4})^{-4} \\ &\Rightarrow i = (1 - \frac{0.05}{4})^{-4} - 1 \\ &\Rightarrow i = 5.1602\%. \end{aligned}$$

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Example. If $i = 3\%$ find $d^{(2)}$.

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Example. If $i = 3\%$ find $d^{(2)}$.

Solution. We solve for $d^{(2)}$ in $1 + i = (1 - \frac{d^{(2)}}{2})^{-2}$:

$$1 + i = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1 + 0.03 = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1.03 = (1 - \frac{d^{(2)}}{2})^{-2}$$
$$\Rightarrow (1.03)^{-\frac{1}{2}} = (1 - \frac{d^{(2)}}{2}) \Rightarrow \frac{d^{(2)}}{2} = 1 - (1.03)^{-\frac{1}{2}}$$
$$\Rightarrow d^{(2)} = 2(1 - (1.03)^{-\frac{1}{2}})$$
$$\Rightarrow d^{(2)} = 2.9341\%$$

- **Force of interest.** The force of interest is the measure of interest at individual moments of time.

- The force of interest at time t, denoted by δ_t, is defined as

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt}\ \ln(A(t))\ .$$

- To find the force of interest, we may use the accumulation function

$$\begin{aligned}\frac{d}{dt}\ \ln(A(t)) = \frac{d}{dt}\ \ln(A(0)a(t)) &= \frac{d}{dt}(\ln(A(0)) + \frac{d}{dt}\ln(a(t)) \\ &= \frac{d}{dt}\ln(a(t))\ .\end{aligned}$$

- $\ln(ab) = \ln(a) + \ln(b)$
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Section 1.6: Continuous Interest and Force of Interest

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Example. Consider the amount function $A(t) = 25(1 + \frac{t}{4})^3$. At what time is the force of interest equal of 0.5.

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Example. Consider the amount function $A(t) = 25(1 + \frac{t}{4})^3$. At what time is the force of interest equal of 0.5.

Solution. The force of interest is

$$\delta_t = \frac{d}{dt} \ln(A(t)) = \frac{d}{dt} \ln \left(25(1 + \frac{t}{4})^3\right) = \frac{d}{dt}\left(\ln 25 + 3 \ln(\frac{4+t}{4})\right) = \frac{d}{dt}\left(3 \ln(4+t) - 3 \ln(4)\right) = \frac{3}{4+t}$$

- $\ln a^r = r \ \ln a$
- $ln(\frac{a}{b}) = \ln(a) - \ln(b)$
- $\frac{d}{dt} \ln 25 = 0$

Since $\delta_t = \frac{1}{2}$, then

$$\frac{3}{4+t} = \frac{1}{2} \Rightarrow 4 + t = 6 \Rightarrow t = 2$$

Section 1.6: Continuous Interest and Force of Interest

The force of interest under simple and compound interest.

Under simple interest: $a(t) = 1 + it$, we have

$$\delta_t = \frac{d}{dt} \ln a(t) = \frac{d}{dt} \ln(1 + it) = \frac{i}{1 + it} .$$

Note: The force of interest is decreasing with t.

Under compound interest: $a(t) = (1 + i)^t$, we have

$$\delta_t = \frac{d}{dt} \ln a(t) = \frac{d}{dt} \ln(1 + i)^t = \frac{d}{dt}\Big(t \ \ln(1 + i)\Big) = \ln(1 + i) .$$

Note: Under compound interest, the force of interest is a constant δ, such that

$$\delta = \ln(1 + i) = -\ln \nu \qquad \text{(Remember } \nu = \frac{1}{1 + i}\text{)}$$

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Finding the accumulation function using the force of interest.

Theorem: For each $t \geq 0$, $a(t) = e^{\int_0^t \delta_s \ ds}$.

Proof. We know that $\delta_s = \frac{d}{ds} \ln a(s)$ and $a(0) = 1$.

$$\int_0^t \delta_s \ ds = \int_0^t \frac{d}{ds} \ln a(s) = \ln a(s)\Big|_0^t = \ln a(t) - \ln a(0) = \ln a(t) \qquad \text{Remember: } \ln(1) = 0$$

$$\Rightarrow \ln a(t) = \int_0^t \delta_s \ ds$$

$$\Rightarrow a(t) = e^{\int_0^t \delta_s \ ds} \qquad \text{Remember: } e^{\ln a(t) = a(t)}$$

Example. A bank account credits interest using a force of interest $\delta_t = \frac{3t^2}{t^3+2}$. A deposit of 100 is made in the account at time $t = 0$. Find the amount of interest earned by the account from the end of the 4th year until the end of the 8th year.

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Solution. First, we want to find

$$a(t) = e^{\int_0^t \delta_s \, ds}$$

$$\int_0^t \delta_s = \int_0^t \frac{3s^2}{s^3+2} \, ds = \ln(s^3+2)\Big|_0^t = \ln(t^3+2) - \ln(2) = \ln\left(\frac{t^3+2}{2}\right)$$

Remember: $\ln \frac{a}{b} = \ln a - \ln b$

Using

$$a(t) = e^{\int_0^t \delta_s \, ds}$$

we have

$$a(t) = e^{\ln\left(\frac{t^3+2}{2}\right)} = \frac{t^3+2}{2}$$

The amount of interest earned in the considered period is

$$100(a(8) - a(4)) = 100\left(\frac{8^3+2}{2} - \frac{4^3+2}{2}\right) = 22400$$

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Example. You deposit 1,000 into an account earning a force of interest of 0.06. How long will it take to triple your money?

Solution. First, $a(t) = k.e^{\int_0^t \delta_s \; ds}$.

$$3000 = 1000e^{\int_0^t 0.06s \; ds}$$

$$\Rightarrow 3 = e^{0.06t} \qquad \text{Note: } \int_0^t 0.06 \; ds = 0.06t\Big|_0^t = 0.06t$$

$$\Rightarrow \ln(3) = 0.06t \; \ln(e) \Rightarrow t = \frac{\ln(3)}{0.06} = 18.3102 \text{ years}$$

Section 1.12: Solving for PV, FV, n, and i with Compound Interest

Example. You deposit 1,000 into an account earning an annual effective rate of 5%, but with interest payable only at the end of each year. After how many years will the account balance be at least 2,000?

Section 1.12: Solving for PV, FV, n, and i with Compound Interest

Example. You deposit 1,000 into an account earning an annual effective rate of 5%, but with interest payable only at the end of each year. After how many years will the account balance be at least 2,000?

Solution. We know, $a(t) = k.(1+i)^t$.

$$2000 = 1000(1+5\%)^t$$
$$2 = (1.05)^t \Rightarrow \ln(2) = t\ln(1.05)$$
$$\Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = 14.2067$$

Note that the interest is not earned until the end of the year, you will not have 2,000 after 14.2067 years, but you will have more than 2,000 at the end of the 15th year. The answer is therefore 15 years.

Exercises

Exercise 1. (Practice Exam 6 Exam FM Page PE6-1: Exercise 2)

The rate at which an investment grows is greatest under which of the following interest scenarios?
A) $d = 0.056$ B) $\delta = 0.057$ C) $d^{(2)} = 0.058$ D) $i^{(4)} = 0.059$ E) $i = 0.060$

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Solution.
A) $d = 0.056$

$$1 + i = \frac{1}{1-d} = \frac{1}{1-0.056} = 1.0593 \Rightarrow i = 5.93\%$$

B) $\delta = 0.057$

$$\delta = \ln(1+i) \Rightarrow 1 + i = e^{\delta} = e^{0.057} = 1.0587 \Rightarrow i = 5.87\%$$

C) $d^{(2)} = 0.058$

$$1 + i = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1 + i = (1 - \frac{0.058}{2})^{-2} = 1.0606 \Rightarrow i = 6.06\%$$

D) $i^{(4)} = 0.059$

$$1 + i = (1 + \frac{i^{(4)}}{4})^{4} = (1 + \frac{0.059}{4})^{4} = 1.0603 \Rightarrow i = 6.03\%$$

E) $i = 0.060$

$$i = 6\%$$

Answer: C

Exercises

Exercise 2. **(Practice Exam 6 Exam FM Page PE6-1: Exercise 11)**

The balance in an account 1.5 years from today will be 100 (assuming no deposits or withdrawals during that time). Find the current balance if the account earns interest based on a nominal rate of discount of 5% convertible quarterly.
A) 86.8 B) 96.4 C) 92.7 D) 92.9 E) 92.2

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A) 86.8 B) 96.4 C) 92.7 D) 92.9 E) 92.2

Solution. The present value at time 0: $k.a^{-1}(t)$

$$a(t) = \left(1 - \frac{d^{(4)}}{4}\right)^{-(4)(1.5)} \qquad \text{Remember: } a(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$$

$$\Rightarrow a(t) = \left(1 - \frac{0.05}{4}\right)^{-6} = 1.07839$$

The present value at time 0: $k.a^{-1}(t) = \frac{100}{1.07839} = 92.73$

Answer: C

Exercises

1. Calculate the present value today of a sum of $20,000 payable in 15 years, assuming an annual interest rate of 5% calculated annually.
2. Calculate the future value, 6 years from now, of a $5,000 deposit made today, assuming an annual interest rate of 5% calculated annually.
3. Given an interest rate of 6%, compute $i_{[2,3]}$ for compound interest.
4. Given an interest rate of 6%, compute $i_{[2,3]}$ for simple interest.
5. Determine the annual effective discount rate that is equivalent to a nominal annual discount rate of 5.5% convertible monthly.
6. Calculate the nominal annual discount rate convertible quarterly that corresponds to an annual effective discount rate of 6.3%.
7. Determine the annual effective discount rate that is equivalent to a nominal discount rate of 7.5% convertible every 4 months ($m = 3$).
8. Determine the nominal annual discount rate convertible monthly that is equivalent to an annual effective discount rate of 6%.
9. Determine the rate of discount convertible quarterly that is equivalent to a nominal interest rate of 8% convertible semi-annually.
10. You wish to accumulate $135,000 in a college fund in 16 years. How much should you deposit now into an account earning an annual effective interest rate of 5.4%?
11. You deposit $1,000 into an account earning an annual effective interest rate of 5.75%. How much will the account hold after 10 years?
12. You deposit $2,500 into an account with a force of interest of 0.07. How long will it take for your investment to triple?

Exercises

13. A man deposited \$3,500 into an account with an annual effective interest rate of 7.5%, with interest paid only at the end of each year. After how many years will the account balance reach at least \$10,000?

14. Alex makes an investment with an initial payment of \$1,750 now and will receive \$5,100 in 7 years. What is the annual effective interest rate Alex will earn?

15. Emma deposited \$1,000 into a bank account that earned 6.5% interest, paid quarterly. What will Emma's balance be after 4.5 years?

16. Michael's investment of \$1,200 accumulates to \$3,500 in 13 years. What nominal rate of interest, convertible monthly, did Michael's investment earn?

17. Sarah deposits \$1,700 into an account at time 0. The account earns a 7.5% annual effective rate. At the end of 3 years, she withdraws \$300. What is the account balance at the end of 4 years?

18. Emma deposits an amount X into an account at time 0 and $4X$ into the same account at time 3. The account balance at time 5 is \$7,500. If the account has earned a 6.4% annual effective rate, what is the value of X?

19. Michael deposits \$2,000 into an account that earns an annual effective rate of 4% during the first 2.5 years, and 6% in all subsequent years. What is the account balance at the end of 4 years?

20. What constant annual effective interest rate would have produced the same 4-year accumulation as in Exercise 19?

21. Determine the effective interest rate over the intervals $[5, 6]$ and $[5.5, 6]$, assuming an annual interest rate of 6.5%, for both (a) compound interest and (b) simple interest.

22. Alex deposits \$2,350 into an account earning a force of interest of 0.065. How long will it take for Alex's account balance to reach \$5,700?

23. Jamie makes an investment where Jamie pays \$11,500 now and receives \$20,000 in 5 years. What nominal rate of interest convertible semi-annually did Jamie earn?

Exercises

24 Alex deposits \$2,400 into an account that earns a nominal annual rate of 7.5%, convertible quarterly, for the first two years. For the next three years, the account earns a nominal rate of 9%, convertible semi-annually.
a) What is Alex's account balance at the end of 5 years?
b) What is the equivalent level nominal rate convertible monthly for this account over the 5-year period?

25 Bruce deposits \$150 into a bank account. His account is credited interest at a nominal rate of 5%, convertible quarterly. At the same time, Peter deposits \$80 into a separate account. Peter's account is credited interest at a force of interest of δ . After 8.25 years, the value of each account is the same. Calculate δ.

26 Sarah invests \$250 into a savings account at time zero, with interest compounded semi-annually at a nominal rate of i. John deposits \$500 into a separate savings account at the same time, earning simple interest at an annual rate of i. During the final six months of the tenth year, both Sarah and John accrue the same amount of interest. Determine the value of i.

27 A loan with an effect annual interest rate of 5% is to be repaid with \$12,000 at the end of each year. If it takes 3 years to pay the loan, assuming compound interest, then the loaned amount at time 0 should be:

28 At what rate of simple interest will \$3500 accumulate to \$4000 in $2\frac{1}{2}$ years?

29 What is the force of interest δ corresponding to the accumulation function $a(t) = (2t+1)^3$?

30 Peter makes deposits of 100 at time 0 and X at time 2. The fund grows at a force of interest

$$\delta_t = \frac{t^2}{100}, \ t > 0$$

The amount of interest earned from time 2 to time 5 is also X. Calculate X.

31 If $i^{(4)} = 9\%$, what is the equivalent $i^{(12)}$?

32 Suppose the accumulation function for your account is $a(t) = 3t + 1$. If you invest \$2000 at time 3 in this fund how much your fund worth at time 8?