Financial Mathematics

Prof. Mohamad Alghamdi

Department of Mathematics

March 2, 2025

Main Content

1. Loan Repayment
2. Calculating the loan balance: the Retrospective and Prospective Method
3. The Amortization Method of Loan Repayment

Section 3.1: Loan Repayment

A **loan** is a form of debt incurred by an individual or other entity. The lender usually a corporation, financial institution, or government advances a sum of money to the borrower.

A **borrower** (also called debtor) receives money by borrowing money with the promise to repay the amount borrowed, or principal, and to pay compensation for borrowing the funds, or interest, to the lender.

The amortization method: all the payments made by the borrower reduce the outstanding balance of the loan. When a loan is paid usually, the total amount of loan payments exceed the loan amount.

The finance charge is the total amount of interest paid (the total payments minus the loan payments).

The simplest way to pay a loan is by unique payment. Suppose that a borrower takes a loan with amount L at time zero and the lender charges an annual effective rate of interest of i. If the borrower pays the loan with a lump sum P at time n, then $P = L(1+i)^n$. The finance charge in this situation is $P - L = L(1+i)^n - L$.

Example: Juan borrows \$35,000 for four years at an annual nominal interest rate of 7.5% convertible monthly. Juan will pay the loan with a unique payment at the end of four years.

(i) Find the amount of this payment.

(ii) Find the finance charge which Juan is charged in this loan.

Section 3.1: Loan Repayment

■ A **loan** is a form of debt incurred by an individual or other entity. The lender usually a corporation, financial institution, or government advances a sum of money to the borrower.

■ A **borrower** (also called debtor) receives money by borrowing money with the promise to repay the amount borrowed, or principal, and to pay compensation for borrowing the funds, or interest, to the lender.

■ **The amortization method:** all the payments made by the borrower reduce the outstanding balance of the loan. When a loan is paid usually, the total amount of loan payments exceed the loan amount.

■ **The finance charge** is the total amount of interest paid (the total payments minus the loan payments).

■ The simplest way to pay a loan is by unique payment. Suppose that a borrower takes a loan with amount L at time zero and the lender charges an annual effective rate of interest of i. If the borrower pays the loan with a lump sum P at time n, then $P = L(1+i)^n$. The finance charge in this situation is $P - L = L(1+i)^n - L$.

Example: Juan borrows \$35,000 for four years at an annual nominal interest rate of 7.5% convertible monthly. Juan will pay the loan with a unique payment at the end of four years.

(i) Find the amount of this payment.

(ii) Find the finance charge which Juan is charged in this loan.

Solution:
(i) The amount of the loan payment is

$$P = 35000(1 + \frac{0.075}{12})^{12\times 4} = 47200.97$$

(ii) The finance charge which Juan is charged in this loan is

$$P - L = L(1+i)^n - L = 47200.97 - 35000 = 12200.97$$

Section 3.1: Loan Repayment

- Under compound interest, the accumulated function: $a(t) = (1+i)^t$.
- Consider a loan L with level payments P to be made at the end of each year for n years at an annual effective interest rate of i%.

Time	0	1	2	...	n
inflow	L	P	P	...	P

The equation of value is the loan: $L = Pa_{\overline{n}|i}$.

- Generally, suppose that a borrower takes a loan of L at time 0 and repays the loan in a series of payments $C_1, \ldots, C_n$ at times $t_1, \ldots, t_n$, where $0 < t_1 < t_2 < \cdots < t_n$. The debtor cashflow is

Time	0	t_1	t_2	...	t_n
inflow	L	C_1	C_2	...	C_n

Since the loan will be repaid, the present value a time zero (or any other time) of this cashflow is zero.

$$L = C_1(1+i)^{-t_1} + C_2(1+i)^{-t_2} + C_3(1+i)^{-t_3} + \cdots + C_n(1+i)^{-t_n} = \sum_{j=1}^{n} C_j(1+i)^{-t_j} = \sum_{j=1}^{n} C_j \nu^{t_j}$$

- Assume that the loan increases with a certain accumulation function $a(t)$, $t \geq 0$. Since the loan will be repaid, the present value a time zero (or any other time) of this cashflow is zero. Hence

$$L = \sum_{j=1}^{n} \frac{C_j}{a(t_j)}$$

The finance charge for this loan is $P - L = \sum_{j=1}^{n} C_j - \sum_{j=1}^{n} \frac{C_j}{a(t_j)} = \sum_{j=1}^{n} \left(C_j - \frac{C_j}{a(t_j)} \right) = \sum_{j=1}^{n} C_j \left(1 - \frac{1}{a(t_j)} \right)$

Section 3.2: Calculating the loan balance

We consider two approaches to compute the balance of the loan: (1) the retrospective method and (2) the prospective method.

(1) Retrospective Method:
The retrospective method is backward looking. It calculates the loan balance as the accumulated value of the loan at the time of evaluation minus the accumulated value of all installments paid up to the time of evaluation.

For the cashflow

Time	0	t_1	t_2	$\dots$	t_n
inflow	L	C_1	C_2	$\dots$	C_n

The outstanding balance of the loan immediately after the kth payment, is

$$B_k = L.a(t_k) - \sum_{j=1}^{k} \frac{a(t_k)}{a(t_j)} C_j$$

Note:

1. If $k = 0$, $B_0 = L$.
2. If $k = n$, $B_n = 0$.

(2) Prospective Method:
The prospective method is forward looking. It calculates the loan balance as the present value of all future payments to be made.

According with the **prospective method**, the outstanding balance of the loan after the kth payment is equal to the present value of the remaining payments.

For the cashflow

Time	0	t_1	t_2	...	t_n
inflow	L	C_1	C_2	...	C_n

The outstanding balance of the loan immediately after the kth payment, is

$$B_k = \sum_{j=k+1}^{n} \frac{a(t_k)}{a(t_j)} C_j$$

Note:

$$\underbrace{L.a(t_k) - \sum_{j=1}^{k} \frac{a(t_k)}{a(t_j)} C_j}_{\text{Retrospective Method}} = \underbrace{\sum_{j=k+1}^{n} \frac{a(t_k)}{a(t_j)} C_j}_{\text{Prospective Method}}$$

Section 3.2: Calculating the loan balance

An inductive relation for the outstanding balance is

$$B_k = B_{k-1}\,\frac{a(t_k)}{a(t_{k-1})} - C_k$$

Previous relation says that the outstanding balance after the kth payment is the accumulation of the previous outstanding balance minus the amount of the payment made.

During the period $[t_{k-1}, t_k]$, the amount of interest accrued is $I_k = B_{k-1}\left(\frac{a(t_k)}{a(t_{k-1})} - 1\right)$

Immediately before the kth payment, the outstanding balance is

$$B_{k-1}+I_k = B_{k-1}+B_{k-1}\left(\frac{a(t_k)}{a(t_{k-1})} - 1\right) = B_{k-1}\frac{a(t_k)}{a(t_{k-1})} \Rightarrow B_{k-1}+I_k-C_k = B_{k-1}\frac{a(t_k)}{a(t_{k-1})}-C_k \Rightarrow B_{k-1}+I_k-C_k = B_k$$

The kth payment C_k can be split as $C_k - I_k + I_k$:
The term I_k is called the interest portion of the kth payment.
The term $C_k - I_k$ is called the principal portion of the $k$the payment. If $C_k - I_k < 0$, then the outstanding balance increases during the kth period:

$$C_k - I_k = C_k - B_{k-1}\left(\frac{a(t_k)}{a(t_{k-1})} - 1\right) = C_k - B_{k-1}\frac{a(t_k)}{a(t_{k-1})} + B_{k-1} = B_{k-1} - \left(B_{k-1}\frac{a(t_k)}{a(t_{k-1})} - C_k\right) = B_{k-1} - B_k$$

Thus, $C_k - I_k = B_{k-1} - B_k$ and this is the reduction on principal made during the the $k$period. The total amount of reduction on principal is equal to the loan amount:

$$\sum_{k=1}^{n}\left(B_{k-1} - B_k\right) = B_0 - B_n = L - 0 = L$$

Section 3.2: Calculating the loan balance

- Under compound interest,

$$L = \sum_{j=1}^{n} C_j (1+i)^{-t_j}$$

- The outstanding balance immediately after the kth payment is

$$B_k = \underbrace{L(1+i)^{t_k} - \sum_{j=1}^{k} C_j (1+i)^{t_k - t_j}}_{\text{Retrospective Method}} = \underbrace{\sum_{j=k+1}^{n} C_j (1+i)^{t_k - t_j}}_{\text{Prospective Method}}$$

- The inductive relation for outstanding balances is

$$B_k = B_{k-1}(1+i)^{t_k - t_{k-1}} - C_k$$

- The amount of interest accrued during the period $[t_{k-1}, t_k]$ is

$$I_k = B_{k-1}\left((1+i)^{t_k - t_{k-1}} - 1\right)$$

- The principal portion of the $k$the payment is

$$C_k - I_k = C_k - B_{k-1}\left((1+i)^{t_k - t_{k-1}} - 1\right) = B_{k-1} - B_k$$

- The finance charge is

$$\sum_{j=1}^{n} C_j - L = \sum_{j=1}^{n} C_j - \sum_{j=1}^{n} C_j (1+i)^{-t_j} = \sum_{j=1}^{n} C_j \left(1 - (1+i)^{-t_j}\right)$$

Section 3.2: Calculating the loan balance

Usually, we consider payments made at equally spaced intervals of time and compound interest. Suppose that a borrower takes a loan L at time 0 and repays the loan in a series of level payments $C_1, \ldots, C_n$ at times $t_0, 2t_0, \ldots, nt_0$. By a change of units, we may assume that $t_0 = 1$. Hence, the debtor cashflow is

Time	0	1	2	3	...	n
inflow	L	$-C_1$	$-C_2$	$-C_3$	...	$-C_n$

Let i be the effective rate of interest per period. Then, we have that $L = \sum_{j=1}^{n} C_j(1+i)^{-j}$

The outstanding balance immediately after the kth payment, is

$$B_k = L(1+i)^k - \sum_{j=1}^{k} C_j(1+i)^{k-j} = \sum_{j=k+1}^{n} C_j(1+i)^{k-j}$$

Note: if repaying a loan is made with a level payment P at the end of periods, then

$$B_k = L(1+i)^k - P \underbrace{s_{\overline{k}|i}}_{\text{the future value at } k} = P \underbrace{a_{\overline{n-k}|i}}_{\text{the present value at } k}$$

The inductive relation for outstanding balances is

$$B_k = B_{k-1}(1+i)^{k-(k-1)} - C_k \Rightarrow B_k = (1+i)B_{k-1} - C_k$$

The amount of interest accrued during the kth year is

$$I_k = B_{k-1}\left((1+i)^{k-(k-1)} - 1\right) = B_{k-1}\left(1+i-1\right) \Rightarrow I_k = iB_{k-1}$$

The principal portion of the kth payment is $C_k - I_k = C_k - iB_{k-1} = (1+i)B_{k-1} - B_k - iB_{k-1} = B_{k-1} - B_k$.

The finance charge is

$$\sum_{j=1}^{n} C_j - L = \sum_{j=1}^{n} C_j\left(1-(1+i)^{-j}\right) = \sum_{j=1}^{n} C_j\left(1-\nu^j\right)$$

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Solution: (i) The number of payments made is $(3)(12) = 36$. we use

$$PV = P.a_{\overline{n \times m}|\frac{i^{(m)}}{m}}$$

where m is number of payments each year.

A nominal rate of interest 8.5% convertible monthly means an interest rate of $\frac{i^{(12)}}{12} = \frac{8.5\%}{12} = 0.0071 = 0.71\%$ per month.
Now,

$$25000 = P.a_{\overline{3 \times 12}|\frac{8.5\%}{12}} = P\frac{1 - (\frac{1}{1.0071})^{36}}{0.0071} \Rightarrow P = 789.1884356$$

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Solution: (i) The number of payments made is $(3)(12) = 36$. we use

$$PV = P.a_{\overline{n \times m}|\,\frac{i^{(m)}}{m}}$$

where m is number of payments each year.

A nominal rate of interest 8.5% convertible monthly means an interest rate of $\frac{i^{(12)}}{12} = \frac{8.5\%}{12} = 0.0071 = 0.71\%$ per month.
Now,

$$25000 = P.a_{\overline{3 \times 12}|\,\frac{8.5\%}{12}} = P\frac{1 - (\frac{1}{1.0071})^{36}}{0.0071} \Rightarrow P = 789.1884356$$

(ii) The total amount of payments made by Roger is $(36)(789.1884356) = 28410.78368$.

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Solution: (i) The number of payments made is $(3)(12) = 36$. we use

$$PV = P.a_{\overline{n\times m}|\,\frac{i^{(m)}}{m}}$$

where m is number of payments each year.

A nominal rate of interest 8.5% convertible monthly means an interest rate of $\frac{i^{(12)}}{12} = \frac{8.5\%}{12} = 0.0071 = 0.71\%$ per month.
Now,

$$25000 = P.a_{\overline{3\times 12}|\,\frac{8.5\%}{12}} = P\frac{1-\left(\frac{1}{1.0071}\right)^{36}}{0.0071} \Rightarrow P = 789.1884356$$

(ii) The total amount of payments made by Roger is $(36)(789.1884356) = 28410.78368$.

(iii) The total interest paid by Roger during the duration of the loan is $28410.78368 - 25000 = 3410.78368$.

Section 3.2: Calculating the loan balance

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.

(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Solution:
(iv) The outstanding loan balance immediately after the 12th payment has been made using the retrospective method:

$$\text{We use } \; B_k = L.(1+i)^k - \sum_{j=1}^{k} C_j(1+i)^{k-j} = L(1+i)^k - Ps_{\overline{k|}i}$$

$$B_{12} = (25000)(1+\frac{0.085}{12})^{12} - (789.1884356)s_{\overline{12|}\frac{8.5\%}{12}} = 17361.71419$$

Section 3.2: Calculating the loan balance

Example: Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.
(i) Find the amount of each monthly payment.

(ii) Find the total amount of payments made by Roger.

(iii) Find the total interest paid by Roger during the duration of the loan.

(iv) Calculate the outstanding loan balance immediately after the 12th payment has been made using the retrospective method.

(v) Calculate the outstanding loan balance immediately after the 12th payment has been made using the prospective method.

Solution:
(iv) The outstanding loan balance immediately after the 12th payment has been made using the retrospective method:

$$\text{We use } \; B_k = L.(1+i)^k - \sum_{j=1}^{k} C_j (1+i)^{k-j} = L(1+i)^k - P s_{\overline{k|}i}$$

$$B_{12} = (25000)(1 + \frac{0.085}{12})^{12} - (789.1884356) s_{\overline{12|}\frac{8.5\%}{12}} = 17361.71419$$

(v) The outstanding loan balance immediately after the 12th payment has been made using the prospective method:

$$\text{We use } \; B_k = \sum_{j=k+1}^{n} C_j (1+i)^{k-j} = P a_{\overline{n-k|}i}$$

$$B_{12} = (789.1884356) a_{\overline{24|}\frac{8.5\%}{12}} = 17361.71419$$

Example: A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.

(i) Calculate the amount of the loan.

(ii) Calculate the outstanding loan balance immediately after the 15th payment has been made by both the prospective and the retrospective method.

(iii) Calculate the amounts of interest and principal paid in the 16th payment.

Example: A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.

(i) Calculate the amount of the loan.

(ii) Calculate the outstanding loan balance immediately after the 15th payment has been made by both the prospective and the retrospective method.

(iii) Calculate the amounts of interest and principal paid in the 16th payment.

Solution: (i) Remember $L = Pa_{\overline{n}|i}$

The cashflow of payments is

Time	1	2	...	10	11	12	...	30
inflow	3000	3000	...	3000	5000	5000	...	5000

The loan amount is

$$L = 3000a_{\overline{10}|4.5\%} + \nu^{10}5000a_{\overline{20}|4.5\%} = 23738.1545 + 41880.8518 = 65619.0063$$

(ii) The outstanding loan balance immediately after the 15th payment using the retrospective method:

$$\text{We use} \quad B_k = L.(1+i)^k - Ps_{\overline{k}|i}$$

$$B_{15} = (65619.0063)(1.045)^{15} - 3000(1.045)^5 s_{\overline{10}|0.045} - 5000s_{\overline{5}|0.045} = 126991.311 - 45940.0337 - 27353.5486 = 53697.7287$$

Example: A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.

(i) Calculate the amount of the loan.

(ii) Calculate the outstanding loan balance immediately after the 15th payment has been made by both the prospective and the retrospective method.

(iii) Calculate the amounts of interest and principal paid in the 16th payment.

Section 3.2: Calculating the loan balance

Example: A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.

(i) Calculate the amount of the loan.

(ii) Calculate the outstanding loan balance immediately after the 15th payment has been made by both the prospective and the retrospective method.

(iii) Calculate the amounts of interest and principal paid in the 16th payment.

Solution:
The outstanding loan balance immediately after the 15th payment using the prospective method:

$$\text{We use } B_k = Pa_{\overline{n-k}|i}$$

$$B_{15} = (5000)a_{\overline{15}|0.045} = 53697.7287$$

(iii) The amount of interest paid in the 16th payment: $I_k = iB_{k-1}$

$$I_{16} = (0.045)B_{15} = 2416.39779$$

The amount of principal paid in the 16th payment: $C_k - I_k$

$$C_{16} - I_{16} = 5000 - 2416.39779 = 2583.60221$$

Amortization Method with Level Payments:
We consider the amortization method of repaying a loan with level payments made at the end of periods of the same length. Let L be the amount borrowed. Let P be the level payment. Let n be the number of payments. Let i be the effective rate of interest per payment period.
The cashflow of payments is

Time	1	2	3	...	n
inflow	P	P	P	...	P

We have that $L = Pa_{\overline{n}|i}$

- The outstanding balance immediately after the kth payment is

$$B_k = \underbrace{L(1+i)^k - Ps_{\overline{k}|i} = P(a_{\overline{n}|i}(1+i)^k - s_{\overline{k}|i})}_{\text{Retrospective Method}} = \underbrace{Pa_{\overline{n-k}|i}}_{\text{Prospective Method}}$$

- The interest portion of the kth payment is $I_k = iB_{k-1} = iPa_{\overline{n+1-k}|i} = P(1-\nu^{n+1-k})$
- The principal reduction of the kth payment is $B_{k-1} - B_k = C_k - I_k = P - iB_{k-1} = P - P(1-\nu^{n+1-k}) = P\nu^{n+1-k}$
- Since $B_{k-1} = Pa_{\overline{n+1-k}|i}$ and $B_{k-1} - B_k = P\nu^{n+1-k}$, we have $B_k = P(a_{\overline{n+1-k}|i} - \nu^{n+1-k})$. The outstanding principal after the kth payment can be found using all these formulas:

$B_k = L(1+i)^k - Ps_{\overline{k}|i} = P(a_{\overline{n}|i}(1+i)^k - s_{\overline{k}|i}) = Pa_{\overline{n-k}|i} = P(a_{\overline{n+1-k}|i} - \nu^{n+1-k})$

The following is the amortization schedule for a loan of $L = Pa_{\overline{n}|i}$ with level payments of P.

Period	Payment	Interest paid	Principal repaid	Outstanding balance
0	-	-	-	$L = Pa_{\overline{n}\|i}$
1	P	$P(1-\nu^n)$	$P\nu^n$	$Pa_{\overline{n-1}\|i}$
2	P	$P(1-\nu^{n-1})$	$P\nu^{n-1}$	$Pa_{\overline{n-2}\|i}$
3	P	$P(1-\nu^{n-2})$	$P\nu^{n-2}$	$Pa_{\overline{n-3}\|i}$
...	...	...	...	...
...	...	...	...	...
k	P	$P(1-\nu^{n+1-k})$	$P\nu^{n+1-k}$	$Pa_{\overline{n-k}\|i}$
...	...	...	...	...
...	...	...	...	...
$n-1$	P	$P(1-\nu^2)$	$P\nu^2$	$Pa_{\overline{1}\|i}$
n	P	$P(1-\nu)$	$P\nu$	0

Remember:

- The interest portion of the kth payment is $I_k = iB_{k-1} = P(1-\nu^{n+1-k})$
- The principal reduction of the kth payment is $B_{k-1} - B_k = \underbrace{C_k}_{=\text{P}} - I_k = P\nu^{n+1-k}$
- The outstanding balance immediately after the kth payment is $B_k = Pa_{\overline{n-k}|i}$

Amortization is the gradual reduction of an amount over time.

The following is the amortization schedule of a loan of \$30,000 at an effective interest rate of 8% for 5 years. It shows how the entire loan balance of 30,000 is amortized over 5 years by the level annual payments of 7,513.69 $L = P\, a_{\overline{n}|i} \Rightarrow P = \frac{30{,}000}{a_{\overline{5}|8\%}}$

Time	Payment	Interest Paid	Principal Paid	Balance
0				30,000.00
1	7,513.69	$\underbrace{2,400.00}_{(iB_{k-1}=0.08\times 30{,}000)}$	$\underbrace{5,113.69}_{(7{,}513.69-2{,}400.00)}$	$\underbrace{24,886.31}_{(30{,}000-5{,}113.69)}$
2	7,513.69	1,990.90	5,522.79	19,363.52
3	7,513.69	1,549.08	5,964.61	13,398.90
4	7,513.69	1,071.91	6,441.78	6,957.12
5	7,513.69	556.57	6,957.12	0.00

Key points:

- The final balance of the loan is 0 (the 5 level payments pay off the loan as intended).
- As the balance declines over time, the amount of interest due in each period decreases and the amount of principal increases.
- It can be seen that the amount of principal repaid increases each year by a factor of $(1+i)$:

$$5,522.79 \times (1+0.08) = 5,964.61$$

$$5,113.69 \times (1+0.08)^2 = 5,964.61 \text{ , etc}$$

Example: A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.
(i) Find the amount of each annual installment.

(ii) Find the finance charge of this loan.

(iii) Find how much interest is accrued in the first year.

(iv) Find how principal is repaid in the first payment.

(v) Find the balance in the loan immediately after the first payment.

Example: A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.
(i) Find the amount of each annual installment.

(ii) Find the finance charge of this loan.

(iii) Find how much interest is accrued in the first year.

(iv) Find how principal is repaid in the first payment.

(v) Find the balance in the loan immediately after the first payment.

Solution: (i) From $L = Pa_{\overline{n}|i}$, we have $100000 = Pa_{\overline{15}|6\%}$ and this implies $P = \frac{100000}{a_{\overline{15}|6\%}} = 10296.2764$

(ii) The finance charge is $(15)(10296.2764) - 100000 = 54444.146$

(iii) The amount of interest accrued in the first year is $I_1 = P(1 - \nu^{n+1-k}) = (10296.2764)(1 - \nu^{15}) = 6000$. OR $I_1 = iB_0$

(iv) The amount of principal repaid in the first year is $C_1 - I_1 = 10296.2764 - 6000 = 4296.2764$

(v) The balance in the loan immediately after the first payment is

$$B_1 = L(1+i) - Ps_{\overline{1}|0.06} = (100000)(1.06) - (10296.2764)s_{\overline{1}|0.06} = 95703.7236$$

$$\text{OR} \quad B_1 = Pa_{\overline{15-1}|6\%} = (10296.2764)\frac{1-(1+6\%)^{-14}}{6\%} = 95703.7236$$

Example: A loan L is being paid with 20 equal annual payments at the end of each year. The principal portion of the 8th payment is 827.65 and the interest portion is 873.81. Find L.

Section 3.3: The Amortization Method of Loan Repayment

Example: A loan L is being paid with 20 equal annual payments at the end of each year. The principal portion of the 8th payment is 827.65 and the interest portion is 873.81. Find L.

Solution: Using $B_{k-1} - B_k = P\nu^{n+1-k}$ and $I_k = iB_{k-1} = P(1 - \nu^{n+1-k})$ where $k = 8$.

Then,

$$B_7 - B_8 = P\nu^{13} = 827.65 \text{ and } iB_7 = P(1 - \nu^{13}) = 873.81$$

Adding the two equations, we get that

$$\begin{aligned} 827.65 &= P\nu^{13} \\ 873.81 &= P(1 - \nu^{13}) \qquad\qquad P(1 - \nu^{13}) = P - P\nu^{13} \\ &- - - - - - - - - \\ 1701.46 &= P \end{aligned}$$

From $827.65 = P\nu^{13}$, we have

$$\nu^{13} = \frac{827.65}{1701.46} \Rightarrow \left(\frac{1}{1+i}\right)^{13} = \frac{827.65}{1701.46} \Rightarrow \frac{1}{1+i} = \left(\frac{827.65}{1701.46}\right)^{\frac{1}{13}} \Rightarrow 1 + i = 1.057 \Rightarrow i = 0.057$$

Now, from $L = Pa_{\overline{n}|i}$, we have

$$L = (1701.46)a_{\overline{20}|0.057} = 20000$$

Section 3.3: The Amortization Method of Loan Repayment

Pay off the loan with interest and principal in equal installments

Suppose that a loan of amount L is paid at the end of each year for n years. At the end of each year two payments are made:

(1) The interest accrued

(2) The principal payment (constant amount): $n\times$ Principal $= L \Rightarrow$ The principal payment $= \frac{L}{n}$

- At the end of j years the outstanding balance is $B_j = \frac{L(n-j)}{n}$

$$B_j = L - \frac{L}{n}j = \frac{Ln - Lj}{n}$$
$$B_j = \frac{L(n-j)}{n}$$

- The interest payment at the end of j years is $I_j = iB_{j-1} = i\,\frac{L(n+1-j)}{n}$.
- The total payment made at the end of the jth year is

$$\underbrace{\frac{L}{n}}_{\text{the principal}} + \underbrace{i\,\frac{L(n+1-j)}{n} = \frac{L}{n}(1+i(n+1-j))}_{\text{the interest}}$$

Payoff the loan with interest and principal in equal installments:

- The principal payment of $\frac{L}{n}$.
- $B_j = \frac{L(n-j)}{n}$
- $I_j = iB_{j-1} = i\frac{L(n+1-j)}{n}$.
- The total payment made at the end of the jth year is $\frac{L}{n} + i\frac{L(n+1-j)}{n} = \frac{L}{n}(1 + i(n+1-j))$

Example: Samuel takes a loan of $175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.

Payoff the loan with interest and principal in equal installments:

- The principal payment of $\frac{L}{n}$.
- $B_j = \frac{L(n-j)}{n}$
- $I_j = iB_{j-1} = i\frac{L(n+1-j)}{n}$.
- The total payment made at the end of the jth year is $\frac{L}{n} + i\frac{L(n+1-j)}{n} = \frac{L}{n}(1+i(n+1-j))$

Example: Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.
(i) Find the amount of each payment of principal.
(ii) Find the outstanding balance owed at the end of the ninth year.
(iii) Find the interest accrued during the tenth year.
(iv) Find the total amount of payments made at the end of the tenth year.
(v) Find the total amount of payments which Samuel makes.

Solution: (i) The annual payment of principal is $\frac{175000}{15} = 11666.67$.
(ii) The outstanding balance owed at the end of the ninth year is $B_9 = \frac{(175000)(15-9)}{15} = 70000$.
(iii) The amount of interest paid at the end of the tenth year is $I_{10} = (0.085)70000 = 5950$.
(iv) The total amount of payments made at the end of the tenth year is $11666.67 + 5950 = 17616.67$.
(v) The interest payment at the end of j years is $i\frac{L(n+1-j)}{n} = (0.085)\frac{(175000)(16-j)}{15}$. Hence, the total interest payments are

$$(0.085)\sum_{j=1}^{15}\frac{(175000)(16-j)}{15} = (0.085)\frac{(175000)}{15}\left(\sum_{j=1}^{15}16 - \sum_{j=1}^{15}j\right) = (0.085)\frac{175000}{15}\left(16\times 15 - \frac{16\times 15}{2}\right) = 119000$$

Remember. $\sum_{j=1}^{n} c = c\times n$ and $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

The total amount of payments which Samuel makes is $119000 + 175000 = 294000$

Example: A housing loan of \$400,000 was to be repaid over 20 years by monthly installments of an annuity–immediate at the nominal rate of 5% per year. After the 24th payment was made, the bank increased the interest rate to 5.5%. (1) If the lender was required to repay the loan within the same period, how much would be the increase in the monthly installment. (2) If the installment remained unchanged, how much longer would it take to pay back the loan?

Section 3.3: The Amortization Method of Loan Repayment

Example: A housing loan of \$400,000 was to be repaid over 20 years by monthly installments of an annuity–immediate at the nominal rate of 5% per year. After the 24th payment was made, the bank increased the interest rate to 5.5%. (1) If the lender was required to repay the loan within the same period, how much would be the increase in the monthly installment. (2) If the installment remained unchanged, how much longer would it take to pay back the loan?

Solution: (1) The amount of the monthly installment: $L = Pa_{\overline{m\times n}|\frac{i^{(m)}}{m}} \Rightarrow 400000 = Pa_{\overline{12\times 20}|\frac{0.05}{12}} \Rightarrow P = 2,639.82$ By the prospective method, after the $k-$th payment the loan is $B_k = Pa_{\overline{n-k}|i}$

After the 24th payment the loan would be $B_{24} = (2,639.82)a_{\overline{240-24}|\frac{0.05}{12}} = 2,639.82 \times 142.241 = 375,490$

After the increase in the rate of interest, if the loan is to be repaid within the same period, the revised monthly installment is

$$L = Pa_{\overline{m\times n}|\frac{i^{(m)}}{m}} \Rightarrow B_{24} = Pa_{\overline{240-24}|\frac{0.055}{12}} \Rightarrow 375,490 = Pa_{\overline{216}|\frac{0.055}{12}} \Rightarrow P = \frac{375,490}{a_{\overline{216}|\frac{0.055}{12}}} = 2,742.26$$

The increase in installment is $2,742.26 - 2,639.82 = 102.44$

(2) Let m be the remaining number of installments if the amount of installment remained unchanged P. Thus,

$$Pa_{\overline{m}|\frac{0.055}{12}} = Pa_{\overline{216}|\frac{0.05}{12}} \Rightarrow \frac{1-\nu^m}{\frac{0.055}{12}} = 142.241 \Rightarrow \frac{1-(1+\frac{0.055}{12})^{-m}}{\frac{0.055}{12}} = 142.241$$

$$\Rightarrow (1+\frac{0.055}{12})^{-m} = 1 - (\frac{0.055}{12})(142.241)$$

$$\Rightarrow -m\ln(1+\frac{0.055}{12}) = \ln\left(1-(\frac{0.055}{12})(142.241)\right)$$

$$\Rightarrow -m = \frac{\ln(0.34806)}{\ln(1+\frac{0.055}{12})} \Rightarrow m = 230.88 \approx 231$$

Thus, it takes $231 - 216 = 15$ months more to pay back the loan.

Example: A housing loan is to be repaid with a 15-year monthly annuity-immediate of \$2,000 at a nominal rate of 6% per year. After 20 payments, the borrower requests for the installments to be stopped for 12 months. (1) Calculate the revised installment when the borrower starts to pay back again, so that the loan period remains unchanged. (2) What is the difference in the interest paid due to the temporary stoppage of installments?

Example: A housing loan is to be repaid with a 15-year monthly annuity-immediate of \$2,000 at a nominal rate of 6% per year. After 20 payments, the borrower requests for the installments to be stopped for 12 months. (1) Calculate the revised installment when the borrower starts to pay back again, so that the loan period remains unchanged. (2) What is the difference in the interest paid due to the temporary stoppage of installments?

Solution: (1) The loan is to be repaid over $15 \times 12 = 180$ payments. After 20 payments, the loan still has 160 installments to be paid.

Using the prospective method $B_k = Pa_{\overline{n-k|}i}$, the balance of the loan after 20 payments is

$$B_{20} = (2000)a_{\overline{180-20|}\frac{6\%}{12}} = 219,910$$

Due to the delay in payments, the loan balance 12 months after the 20th payment is

$$a(t) = k(1 + \frac{i^{(m)}}{m})^{m \times n} = 219,910(1.005)^{12} = 233,473$$

This amount of the loan has to be repaid with a 148-payment (i.e. $180 - 20 - 12 = 148$) annuity-immediate.

Hence, the revised installment is

$$L = Pa_{\overline{148|}\frac{6\%}{12}} \Rightarrow P = \frac{233,473}{a_{\overline{148|}\frac{6\%}{12}}} = 2,236.31$$

(2) The difference in the interest paid is $2,236.31 \times 148 - 2,000 \times 160 = 10,973$.

Example: A man borrows a housing loan of \$500,000 from Bank A to be repaid by monthly installments over 20 years at nominal rate of interest of 4% per year. After 24 installments Bank B offers the man a loan at rate of interest of 3.5% to be repaid over the same period. However, if the man wants to re-finance the loan he has to pay Bank A a penalty equal to 1.5% of the outstanding balance. If there are no other re-financing costs, should the man re-finance the loan?

Example: A man borrows a housing loan of $500,000 from Bank A to be repaid by monthly installments over 20 years at nominal rate of interest of 4% per year. After 24 installments Bank B offers the man a loan at rate of interest of 3.5% to be repaid over the same period. However, if the man wants to re-finance the loan he has to pay Bank A a penalty equal to 1.5% of the outstanding balance. If there are no other re-financing costs, should the man re-finance the loan?

Solution: The monthly installment paid to Bank A is

$$L = Pa_{\overline{12\times 20|}\,\frac{4\%}{12}} \Rightarrow P = \frac{500,000}{a_{\overline{12\times 20|}\,\frac{4\%}{12}}} = 3,029.94$$

The outstanding balance after paying the 24th installment is

$$B_k = Pa_{\overline{n-k|}\,i} \Rightarrow B_{24} = (3,029.94)a_{\overline{240-24|}\,\frac{4\%}{12}} = 3,029.94 \times 153.80 = 466,004$$

If the man re-finances with Bank B, he needs to borrow

$$\underbrace{466,004}_{\text{outstanding balance}} + \underbrace{466,004 \times 1.5\%}_{\text{penalty paid to Bank A}} = 472,994$$

The monthly installment with Bank B is

$$L = Pa_{\overline{216|}\,\frac{3.5\%}{12}} \Rightarrow P = \frac{472,994}{a_{\overline{216|}\,\frac{3.5\%}{12}}} = 2,954.56$$

As this is less than the installments of $3,029.94 he pays to Bank A, he should re-finance.

Example: Construct an amortization schedule of a loan of \$5,000 to be repaid over 6 years with a 6-payment annuity-immediate at effective rate of interest of 6% per year.

Example: Construct an amortization schedule of a loan of $5,000 to be repaid over 6 years with a 6-payment annuity-immediate at effective rate of interest of 6% per year.

Solution: The annual payment is

$$L = Pa_{\overline{n}|i} \Rightarrow 5,000 = Pa_{\overline{6}|6\%} \Rightarrow P = \frac{5,000}{a_{\overline{6}|6\%}} = 1,016.81$$

Time	Installment	Interest Paid	Principal Paid	Balance
0				5,000
1	1,016.81	$\underbrace{300}_{(iB_{k-1}=0.06\times 5,000)}$	$\underbrace{716.81}_{(1,016.81-300)}$	$\underbrace{4,283.19}_{(5,000-716.81)}$
2	1,016.81	256.99	759.82	3,523.37
3	1,016.81	211.40	805.41	2,717.96
4	1,016.81	163.08	853.73	1,864.23
5	1,016.81	111.85	904.96	959.26
6	1,016.81	57.55	959.26	0.00
Total	6,100.86	1,100.86	5,000.00	

Note: *The results are calculated to two decimal places.*

Example: A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 2% each year. Find the loan balance after the 4th payment.

Example: A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 2% each year. Find the loan balance after the 4th payment.

Solution: The payments are

$$100, 100(1.02), 100(1.02)^2, 100(1.02)^3, 100(1.02)^4, \ldots, 100(1.02)^9$$

After the 4th payment the remaining payments are

$$100(1.02)^4, 100(1.02)^5, 100(1.02)^6, 100(1.02)^7, 100(1.02)^8, 100(1.02)^9$$

Using the prospective method ($B_4 = Pa^{g}_{\overline{10-4}|i}$), the balance immediately after the fourth payment is

$$\begin{aligned} B_4 &= 100(1.02)^4 a^{0.02}_{\overline{6}|10\%} \\ &= 100(1.02)^4 \frac{1-\left(\frac{1.02}{1.1}\right)^6}{0.1-0.02} \\ &= 492.93 \end{aligned}$$

Remember: $a^{g}_{\overline{n}|} = \dfrac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$

Exercise: A loan at an 8% annual effective rate has an initial payment of 1,000, and 9 further payments. The payment amount decreases by 2% each year. Find the loan balance immediately after the 3rd payment. Answer: 4,644.38

Example: A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 10 each year. Find the loan balance after the 4th payment.

Example: A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 10 each year. Find the loan balance after the 4th payment.

Solution: Payments are $100, 110, 120, 130, 140, \ldots, 190$.

Immediately after the 4th payment the remaining payments are

$$140, 150, 160, 170, 180, 190$$

Using the prospective method ($B_4 = Pa_{\overline{10-4|}i}$), the balance immediately after the 4th payment is the present value of those remaining payments.

$$\begin{aligned} B_4 &= 140a_{\overline{6|}0.10} + (10)\left(\frac{a_{\overline{6|}0.10} - 6\nu^6}{0.10}\right) \quad \text{Remember: } PV = Pa_{\overline{n|}} + Q\left(\frac{a_{\overline{n|}} - n\nu^n}{i}\right) \\ &= 140(4.355) + 10\left(\frac{4.355 - 6(0.5645)}{0.10}\right) = 706.57 \end{aligned}$$

Exercise: A loan at an 8% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount decreases by 5 each year. Calculate the loan balance immediately after the 6th payment. Answer: 208.60

Example: A borrower wants to borrow 30,000 at 8% for 5 years, but would like to pay only 5,000 for the first 2 years and then catch up with larger payments in the final 3 years of the loan. What is the payment amount for the final 3 years?

Example: A borrower wants to borrow 30,000 at 8% for 5 years, but would like to pay only 5,000 for the first 2 years and then catch up with larger payments in the final 3 years of the loan. What is the payment amount for the final 3 years?

Solution:
The amount of interest paid at the end of the first year: $I_1 = 30,000 \times 0.08 = 2,400$
If the payment for the first year is 5000, then the principal payment 5000-2400=2600, so the loan balance is 30,000-2600=27,400.

The amount of interest paid at the end of the second year: $I_2 = 27,400 \times 0.08 = 2,192$
If the payment for the second year is 5000, then the principal payment 5000-2192=2808, so the loan balance is 27,400-2808=24,592.

The outstanding loan balance after 2 years of the payment 5,000 each year is 24,592. The borrower can pay off the 24,592 balance in 3 years by making larger payments:

$$L = Pa_{\overline{3}|i} \Rightarrow P = \frac{24,592}{a_{\overline{3}|i}} = 9,542.52$$

Year	Payment	Interest Paid	Principal Paid	Balance
0				30,000.00
1	5,000.00	2,400.00	2,600.00	27,400.00
2	5,000.00	2,192.00	2,808.00	24,592.00
3	9,542.52	1,967.36	7,575.16	17,016.84
4	9,542.52	1,361.35	8,181.17	8,835.67
5	9,542.52	706.85	8,835.67	0.00

Review: Loan Repayment

Lectures: class notes and slides
Examples from slides;1,2,3,4,5,6,8
Examples from notes: 5.1, 5.2, and 5.3
Examples from Actex: 3.10, 3.12, 3.16
Table(3.17) from Actex
Exercises from Actex: 3.11, 3.13, 3.18
Balloon Payment Exercise
+Qs on Loans Bonds from 2018 sample:
Exam:6,7,12,13,21,23,26,33,34,40,41,42,45,47,48,49,59,60,61,64,67,69,71,72,81,82,87,88,89,90, 92,93,95,96,97,98.