Financial Mathematics

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Main Content

1. Amount and accumulation functions.
2. Simple interest.
3. Compound interest
4. Present value and Discount
5. Nominal rates of interest and Discount
6. Force of interest

Section 1.1: Amount and accumulation functions

- **Principal:** The initial of amount of money (capital) invested.
- **Time:** The time from date of investment.
- **Measurement period (Period):** The unit in which time is measured (days, months, years, decades, etc). The most common measurement is one year and this will be assumed unless stated otherwise.
- **Amount of interest (Interest) I:** Amount of interest is the difference between the accumulated value (final balance) and the principal (invested amount):

$$I = \text{final balance} - \text{invested amount}$$

- **Effective rate of interest i:** is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period:

$$i = \frac{\text{final balance} - \text{invested amount}}{\text{invested amount}}$$

Note:

- The effective rate of interest is expressed in percentage, for example, $i = 8\%$.
- If invested amount is k, then

$$i = \frac{I}{k} \Rightarrow I = i\, k$$

- **Accumulated value:** The total amount received after a period of time.
- **Accumulation function $a(t)$:** gives the accumulated value at time $t \geq 0$ of an original investment of 1 where $a(0) = 1$.
- **Amount function $A(t)$:** gives the accumulated value at time $t \geq 0$ of an original investment of k:

$$A(t) = k\ .\ a(t) \quad \text{or} \quad A(t) = A(0)\ .\ a(t) \quad \text{where} \quad A(0) = k$$

Notes:

- The accumulation function $a(t)$ is a special case of the amount function $A(t)$ for which $k = 1$.
- For each $t \geq 0$, $A(t) > 0$.
- The amount function $A(t)$ is nondecreasing.

Example: Simon invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.
(1) Find the amount of interest earned by Simon in those 6 months.

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- Suppose that an amount $A(0)$ of money is invested at time 0. Then,
- $A(0)$ is the principal.
- $A(t)$ is the value at time t of the initial investment $A(0)$.
- The amount of interest earned over the period $[s, t]$ is $A(t) - A(s)$.
- The effective rate of interest earned in the period $[s, t]$ is $\frac{A(t)-A(s)}{A(s)}$.
- The effective rate of interest earned in the first period is

$$i = \frac{A(1) - A(0)}{A(0)} = \frac{I}{A(0)} \Rightarrow I = ik,$$

where $A(0)$ is the amount of money invested at time zero (principal).

- The effective rate of interest earned in the nth period is

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}, \quad \text{for integer } n \geq 1$$

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Example: Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)(1 + \frac{t}{40})$, where t is the number of years after March 1, 2008.
(1) Find the balance in Jessicas account on October 1, 2008.

(2) Find the amount of interest earned in those 7 months.

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$$\frac{A(7/12) - A(0)}{A(0)} = \frac{72.917}{5000} = 0.0145834 = 1.45834\%$$

Section 1.1: Amount and accumulation functions

Cashflow: a series of payments (deposits/withdrawals) made at different times.

The payments can be either made by the individual or to the individual.

- An inflow is payment to the individual and represented by positive numbers.
- An outflow is a payment by the individual and represented by negative numbers.

Consider a situation in which an investor makes deposits or contributions into an investment of $C_0, C_1, \ldots, C_n$ at times 0, 1, 2, $\ldots$, n. Then,

- If $C_t > 0$, there is a net cash flow into the investment at time t.
- If $C_t < 0$, there is a net cash flow out the investment at time t

If deposits/withdrawals are made according with the table

Investments	C_1	C_2	$\ldots$	C_n
Time (in periods)	t_1	t_2	$\ldots$	t_n

Rule 1: Proportionality. If an investment strategy follows the amount function $A(t)$, $t > 0$, an investment of \$k made at time 0 with the previous investment strategy, has a value of \$ $k\frac{A(t)}{A(0)}$ at time t.

Recall: The accumulation function $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.

- By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.
- Knowing the value function $a(t)$ and the principal $A(0)$, we can find the amount function $A(t)$ using the formula $A(t) = A(0)a(t)$.

$$\text{This implies } \; A(t) = A(0)a(t) \Rightarrow a(t) = \frac{A(t)}{A(0)} \Rightarrow k.a(t) = k.\frac{A(t)}{A(0)}$$

Notes:

- Investing $A(0)$ at time zero, we get $A(t)$ at time t.
- Investing 1 at time zero, we get $\frac{A(t)}{A(0)}$ at time t.

Present value (PF): The present value at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.

Note. Let x be the amount which need to invest at time zero to get a balance of k at time t. So, we have

$$k = x\frac{A(t)}{A(0)} \Rightarrow x = k\frac{A(0)}{A(t)} .$$

Hence, the present value at time 0 of a balance of k had at time t is $k\frac{A(0)}{A(t)}$.

Accumulated value or future value (FV): The accumulated value at certain time in the future of a cashflow is the amount obtained by investing at time 0.

Note. If we invest k at time zero, we get \$ $k\frac{A(t)}{A(0)}$ at time t. We say that: the accumulated value at time t (sometimes we say: the present value at time t) of a deposit of k made at time zero is \$ $k\frac{A(t)}{A(0)}$.

Summary. if we know the value of an investment as of a particular date and we want to find its value as of an earlier date, we are calculating a present value as of the earlier date. And if we want to find the value as of a later date, then we are calculating a future value (or an accumulated value) as of that later date.

Using the accumulation function $a(t)$, $t \geq 0$, we have:

The accumulated value at time t (the balance or the present value at time t) of a deposit of k made at time zero is $ka(t) = k\frac{A(t)}{A(0)}$.

The present value at time 0 of a balance of k had at time t is $\frac{k}{a(t)} = k\frac{A(0)}{A(t)}$.

Example: The accumulation function of a fund is $a(t) = (1.03)^{2t}, \ t \geq 0$.
(1) Amanda invests \$5000 at time zero in this fund. Find the balance into Amandas fund at time 2.5 years.

(2) How much money does Kevin need to invest into the fund at time 0 to accumulate \$10000 at time 3?

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Solution:
(1) The balance into Amandas fund at time 2.5 years is

$$ka(2.5) = (5000)(1.03)^{2(2.5)} = 5796.370371$$

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$$\frac{10000}{a(3)} = \frac{10000}{(1.03)^{2(3)}} = 8374.842567$$

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Rule 2: Grows-depends-on-balance rule.

If an investment follows the amount function $A(t), \ t \geq 0$, the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.

If an account has a balance of k at time t and no deposits/withdrawals are made in the future, then the future balance in this account does not depend on how the balance of k at time t was attained.

In particular, the following two accounts have the same balance for times bigger than s:
1. An account where a unique deposit of k is made at time s.
2. An account where a unique deposit of $\frac{k}{A(s)}$ is made at time zero.

Time	0	s	t
Contribution		k	$k\frac{A(t)}{A(s)}$
	$\frac{k}{A(s)}$	k	$k\frac{A(t)}{A(s)}$

Theorem. If an investment follows the amount function $A(.)$, the present value at time t of a deposit of \$k made at time s is \$ $k\frac{A(t)}{A(s)} = k\frac{a(t)}{a(s)}$.

Notes.

- The present value at time t (accumulated value) of an investment of $A(s)$ made at time s is $A(t)$.

- The following three accounts have the same balance at any time bigger than s:

1. An account where a unique deposit of $A(0)$ is made at time zero.

2. An account where a unique deposit of $A(s)$ is made at time s.

3. An account where a unique deposit of $A(t)$ is made at time t.

Time	0	s	t
Contribution		$A(s)$	$A(s)\frac{A(t)}{A(s)} = A(t)$
	$A(0)$	$A(0)\frac{A(s)}{A(0)} = A(s)$	$A(s)\frac{A(t)}{A(s)} = A(t)$
			$A(t)$

Remember.

- The present value at time t of an investment of k made at time s is $k\frac{A(t)}{A(s)}$, i.e. k at time s is equivalent to $k\frac{A(t)}{A(s)}$ at time t.

- The present value at time s of a balance of k had at time t is $k\frac{A(s)}{A(t)}$ i.e. k at time t is equivalent to $k\frac{A(s)}{A(t)}$ at time s.

Example: The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}, \ t > 0$.
(1) Michael invests \$3500 into the fund at time 1. Find the value of Michaels fund account at time 4.

(2) How much money needs Jason to invest at time 2 to accumulate \$700 at time 4.

Example: The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}, \; t > 0$.
(1) Michael invests $3500 into the fund at time 1. Find the value of Michaels fund account at time 4.

(2) How much money needs Jason to invest at time 2 to accumulate $700 at time 4.

Solution:
(1) The value of Michaels account at time 4 is

$$3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000$$

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(1) Michael invests $3500 into the fund at time 1. Find the value of Michaels fund account at time 4.

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(1) The value of Michaels account at time 4 is

$$3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000$$

(2) To accumulate $700 at time 4, Jason needs to invest at time 2,

$$700\frac{a(2)}{a(4)} = 700\frac{1+\frac{2}{20}}{1+\frac{4}{20}} = 700\frac{1.1}{1.2} = 641.67$$

Section 1.1: Amount and accumulation functions

Theorem: Present value of a cashflow. If an investment account follows the amount function $A(t)$, $t > 0$, and $0 \leq t_1 < t_2 < \cdots < t_n$

Time	t_1	t_2	...	t_n
Deposits	C_1	C_2	...	C_n

- The future value (present value at time t) of the cashflow: $V(t) = \sum_{j=1}^{n} C_j \frac{A(t)}{A(t_j)} = \sum_{j=0}^{n} C_j \frac{a(t)}{a(t_j)}$

- The present value at time zero: $V(t) = \sum_{j=1}^{n} C_j \frac{1}{a(t_j)}$

Proof.

Time	t_1	t_2	t_3	...	t_n
Balance before deposit	0	$C_1 \frac{a(t_2)}{a(t_1)} = \sum_{j=1}^{1} C_j \frac{a(t_2)}{a(t_j)}$	$\frac{a(t_3)}{a(t_2)} \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)} = \sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)}$	...	$\frac{a(t_n)}{a(t_{n-1})} \sum_{j=1}^{n-1} C_j \frac{a(t_{n-1})}{a(t_j)} = \sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)}$
Balance after deposit	C_1	$\sum_{j=1}^{1} C_j \frac{a(t_2)}{a(t_j)} + \underbrace{C_2}_{C_2 \frac{a(t_2)}{a(t_2)}} = \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)}$	$\sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)} + \underbrace{C_3}_{C_3 \frac{a(t_3)}{a(t_3)}} = \sum_{j=1}^{3} C_j \frac{a(t_3)}{a(t_j)}$	...	$\sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)} + \underbrace{C_n}_{C_n \frac{a(t_n)}{a(t_n)}} = \sum_{j=1}^{n} C_j \frac{a(t_n)}{a(t_j)}$

Example: The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}, \ t > 0$. Jared invests \$1000 into the fund at time 1 and he withdraws \$500 at time 3. Find the value of Jareds fund account at time 5.

Example: The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$. Jared invests \$1000 into the fund at time 1 and he withdraws \$500 at time 3. Find the value of Jareds fund account at time 5.

Solution: The cashflow is

Deposits	1000	-500
Time	1	3

The value of Jareds account at time 5 is

Remember. Present value $= k\frac{A(t)}{A(s)} = k\frac{a(t)}{a(s)}$

$$1000\frac{a(5)}{a(1)} - 500\frac{a(5)}{a(3)} = 1000\frac{1+\frac{5}{20}}{1+\frac{1}{20}} - 500\frac{1+\frac{5}{20}}{1+\frac{3}{20}} = 1000\frac{1.20}{1.05} - 500\frac{1.20}{1.15} = 1190.48 - 543.48 = 647.00$$

Section 1.2: Simple Interest

■ **Simple interest (S.I)** is the investment interest earned during each period so that the interest is constant.

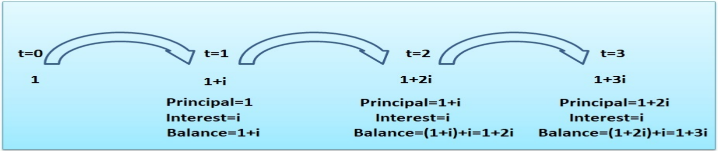
If i is the effective annual rate of simple interest, from investing \$1, we have

- At the end of the first period, the accumulated value (balance): $1 + i$.
- At the end of the second period, the accumulated value (balance): $1 + 2i$.
- At the end of the third period, the accumulated value (balance): $1 + 3i$, etc.

Note

(1) $i = \frac{I}{k} \Rightarrow I = ik$ (2)
Balance at $t=$

Amount at$(t-1)+$Interest



■ Thus, if i is the effective annual rate of simple interest, the investment of 1 at time zero, we have

- The amount of interest at time t years is it.
- The accumulation function (balance) at time t years: $a(t) = 1 + it$ for integer $t \geq 0$

■ In general, we have a linear accumulation function as follows:

If i is the effective annual rate of simple interest, the investment of k at time zero, then

- The amount of interest at time t years: kit.
- The balance at time t years: $k + kit = k(1 + it)$, i.e

- If i is the effective annual rate of simple interest, the investment of k at time s, then
- The amount of interest at time t years ($t > s$): $ki(t-s)$. Remember. $I = ikt$ from $t = 0$
- The balance at time t years: $k + ki(t-s) = k\big(1 + i(t-s)\big)$, i.e

$$\text{The balance in the period}[s, t] \text{ is } \; A(t-s) = k\big(1 + i(t-s)\big) \; \text{ for integer } \; t \geq 0$$

Note: To find the amount of deposit at time s to get a balance of \$ x in time t where $s < t$, we use the above formula as follows:

$$A(t-s) = x \Rightarrow k\ \big(1 + i(t-s)\big) = x \Rightarrow k = \frac{x}{\big(1 + i(t-s)\big)}$$

so, we need to make a deposit of $\frac{x}{\left(1+i(t-s)\right)}$

Example: John invest \$ 2000 for four years if the rate of simple interest is 8% annum, find the following:

(1) The accumulation value,

(2) The amount of interest.

- If i is the effective annual rate of simple interest, the investment of k at time s, then
- The amount of interest at time t years ($t > s$): $ki(t-s)$. Remember. $I = ikt$ from $t = 0$
- The balance at time t years: $k + ki(t-s) = k\big(1 + i(t-s)\big)$, i.e

$$\text{The balance in the period}[s,t] \text{ is } \; A(t-s) = k\big(1+i(t-s)\big) \; \text{ for integer } \; t \geq 0$$

Note: To find the amount of deposit at time s to get a balance of \$ x in time t where $s < t$, we use the above formula as follows:

$$A(t-s) = x \Rightarrow k\ \big(1+i(t-s)\big) = x \Rightarrow k = \frac{x}{\big(1+i(t-s)\big)}$$

so, we need to make a deposit of $\frac{x}{\big(1+i(t-s)\big)}$

Example: John invest \$ 2000 for four years if the rate of simple interest is 8% annum, find the following:

(1) The accumulation value,

(2) The amount of interest.

Solution:

(1) The accumulation value: $A(t) = k(1+it)$

$$A(4) = 2000(1+0.08(4)) = 2640$$

- If i is the effective annual rate of simple interest, the investment of k at time s, then
- The amount of interest at time t years ($t > s$): $ki(t-s)$. Remember. $I = ikt$ from $t = 0$
- The balance at time t years: $k + ki(t-s) = k\big(1 + i(t-s)\big)$, i.e

$$\text{The balance in the period}[s,t] \text{ is } \; A(t-s) = k\big(1+i(t-s)\big) \; \text{ for integer } \; t \geq 0$$

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Solution:

(1) The accumulation value: $A(t) = k(1 + it)$

$$A(4) = 2000(1 + 0.08(4)) = 2640$$

(2) The amount of interest: $I = \text{Balance} - \text{invested amount}$: $I = 2640 - 2000 = 640$

OR use the formula: $I = kit \Rightarrow I = (2000)(8\%)(4) = 640$

Section 1.3: Compound Interest

Compound interest is interest that is automatically reinvested. Meaning that the total investment of principal and interest earned to date is kept invested at all times.

Now, if you invest 1, then

- At the end of the first period, the accumulated value (balance): $1 + i$.

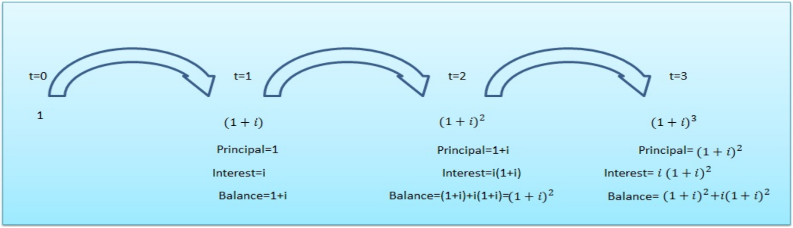
- At the end of the second period, the accumulated value (balance): $(1 + i)^2$.

The balance $1 + i$ can be considered as principal at the beginning of the second period and will earn interest of $i(1 + i)$. The accumulated value (the balance) at the end of the second period is $(1 + i) + i(1 + i) = (1 + i)^2$.

- At the end of the third period, the accumulated value (balance): $(1 + i)^3$, etc.

The balance $(1 + i)^2$ can be considered as principal at the beginning of the third period and will earn interest of $i(1 + i)^2$. The accumulated value (the balance) at the end of the third period is $(1 + i)^2 + i(1 + i)^2 = (1 + i)^3$.

Remember. The interest $I = ik$ and the balance (B) at time t is the $I+$ the balance at time $(t - 1)$



- The accumulated value under the compound interest

$$a(t) = (1 + i)^t \quad \text{for integer } t \geq 0 \quad \text{where } i \text{ is the effective rate of interest}$$

- The amount function is

$$A(t) = A(0)\ (1+i)^t \quad \text{for integer} \quad t \geq 0 \quad \text{where} \quad A(0) \quad \text{is amount of money invested at time 0}$$

Note. The effective rate of interest over a certain period of time depends only on the length of this period, i.e. for $0 \leq s \leq t$,

$$\frac{A(t) - A(s)}{A(s)} = \frac{A(0)(1+i)^t - A(0)(1+i)^s}{A(0)(1+i)^s} = (1+i)^{t-s} - 1 = \frac{A(t-s) - A(0)}{A(0)}$$

where i is the effective annual rate of interest.

Example: John invest \$ 2000 for four years if the rate of compound interest is 8% annum, find the following:

(1) the accumulation value,

(2) the amount of interest.

Section 1.3: Compound Interest

The amount function is

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(1) the accumulation value,

(2) the amount of interest.

Solution:
(1) The accumulation value: $A(t) = k(1+i)^t$

$$A(4) = 2000(1+0.08)^4 = 2720.98$$

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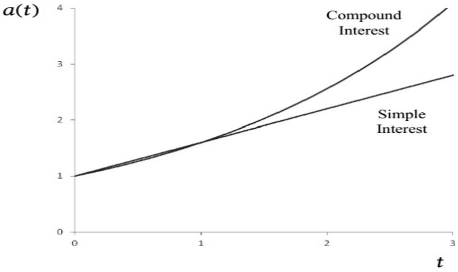
Solution:
(1) The accumulation value: $A(t) = k(1+i)^t$

$$A(4) = 2000(1+0.08)^4 = 2720.98$$

(2) the amount of interest: $I = \text{Balance} - \text{invested amount}$

$$I = 2720.98 - 2000 = 720.98$$

The following graph compares the growth of 2 investments: simple and compound interest. In each case, an amount of 1 is invested at time 0. One investment earns simple interest at 60% annual rate; the other earns compound interest at a 60% annual rate.



Notes:

- for $t = 1$, both investments are equal (they have the same value).
- For $0 \leq t < 1$, the investment at simple interest has a larger value than the investment earning compound interest.
- For $t > 1$, the investment at compound interest has a larger value than the investment at simple interest. Also, after the first year, the compound interest grows much faster than the simple-interest investment.

Section 1.4: Present value and Discount

- **Under compound interest with effective annual rate of interest i:**

- If you invest \$ k for 1 year, the balance is $k(1+i)$. The quantity $(1+i)$ is called **the interest factor.**

- If you invest \$ k for t years, the balance is $k(1+i)^t$. The quantity $(1+i)^t$ is called **the t–year interest factor.**

- To determine how much a person must invest so that the balance will be \$1 at the end of one period:

$$1 = k(1+i) \Rightarrow k = \frac{1}{1+i}$$

- The symbol $\nu = \frac{1}{1+i}$ is called **the discount factor.**

- To determine how much a person must invest so that the balance will be \$1 at the end of t years:

$$1 = k(1+i)^t \Rightarrow k = \frac{1}{(1+i)^t}$$

- If you invested $\frac{1}{(1+i)^t}$ in t years ago in account earning compound interest, then the current balance is \$ 1.

- The quantity $\nu^t = \frac{1}{(1+i)^t}$ is called **the t–year discount.**

Summary:

Under simple interest:

The quantity $a(t) = 1 + it$ is called accumulated value of 1 at the end of t periods. The quantity $k\ a(t) = k(1 + it)$ is called accumulated value of k at the end of t periods (The total amount received after t periods if we deposit k).

The quantity $a^{-1}(t) = \frac{1}{1+it}$ is called the present value t period in the past (or discounted value) of 1 to be paid at the end of t periods. The quantity $k\ a^{-1}(t) = \frac{k}{1+it}$ is called the present value t periods in the past (or discounted value) of k to be paid at the end of t periods (The present value t years in the past i.e. the amount of money which will accumulate to get k over t periods) .

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Example: Find the amount which must be invested at a rate of simple interest of 9 % per annum in order to accumulate $ 1000 at the end of three years.

Section 1.4: Present value and Discount

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Example: Find the amount which must be invested at a rate of simple interest of 9 % per annum in order to accumulate \$ 1000 at the end of three years.

Solution: $k\ a^{-1}(t) = k\ \frac{1}{1+it} = 1000\ \frac{1}{1+(0.09)(3)} = \$\ 787.40$

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Example: Find the amount which must be invested at a rate of compound interest of 9 % per annum in order to accumulate \$ 1000 at the end of three years.

Solution: $k\ \nu^t = k\ \frac{1}{(1+i)^t} = 1000\ \frac{1}{(1+0.09)^3} = \$\ 772.18$

The effective rate of interest and Discount:

- Recall, the effective rate of interest is a measure of interest paid at **the end of the period**.
- The effective rate of discount, denoted by d, as a measure of interest paid at **the beginning of the period**. Sometimes called the amount of discount or just discount.

Further explanation: Take the following two cases:

Case 1: You borrow \$ 100 from the bank for one year at an effective rate of interest 6%. The bank will give \$ 100 and at the end of the year, you will repay the bank the original loan \$ 100 plus interest \$ 6, the total is \$ 106.

Case 2: You borrow \$ 100 from the bank for one year at an effective rate of discount 6 % in advance. The bank will give \$ 94 and at the end of the year, you will repay the bank the original loan \$ 94 plus interest \$ 6, the total is \$ 100.

Notes:

In both cases, you paid \$ 6 interest. However, in case 1, the interest is paid at the end of the year and you had the use of \$ 100 for the year. In case 2, the interest is paid at the beginning of the year and you had the use of \$ 94 for the year.

Interest paid at the end of the period on the balance at the beginning of the period: $i = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Principal}(P)} \Rightarrow i = \frac{I}{P}$

Discount paid at the beginning of the period on the balance at the end of the period: $d = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Balance}(B)} \Rightarrow d = \frac{I}{B}$

Example: Peter invests \$ 738 in a bank account. One year later, his bank account is \$ 765.

(1) Find the effective annual interest rate earned by Peter in that year.

(2) Find the effective annual discount rate earned by Peter in that year.

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- Recall, the effective rate of interest is a measure of interest paid at **the end of the period**.
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Notes:

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- Interest paid at the end of the period on the balance at the beginning of the period: $i = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Principal}(P)} \Rightarrow i = \frac{I}{P}$
- Discount paid at the beginning of the period on the balance at the end of the period: $d = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Balance}(B)} \Rightarrow d = \frac{I}{B}$

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(1) Find the effective annual interest rate earned by Peter in that year.

(2) Find the effective annual discount rate earned by Peter in that year.

Solution: Peter earns an interest amount of $I = 765 - 738 = 27$.

(1) The effective annual interest rate earned by Peter is $i = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Principal}(P)} = \frac{27}{738} = 3.658537$ %.

(2) The effective annual discount rate earned by Peter is $d = \frac{\text{Balance}(B) - \text{Principal}(P)}{\text{Balance}(B)} = \frac{27}{765} = 3.529412$ %.

Section 1.4: Present value and Discount

Notes:

- The effective rate of interest in the n–th year is $i_n = \frac{a(n)-a(n-1)}{a(n-1)}$.
- The effective rate of discount in the n–th year is $d_n = \frac{a(n)-a(n-1)}{a(n)}$.

Remember:

- The quantity $(1+i)$ is called the interest factor.
- The quantity $\nu = \frac{1}{(1+i)}$ is called the discount factor.
- Under the accumulation function $a(t)$, we have
- The $n-$th year interest factor is $\frac{a(n)}{a(n-1)} = \frac{(1+i)^n}{(1+i)^{n-1}} = (1+i)$.
- The $n-$th year discount factor is $\nu_n = \frac{a(n-1)}{a(n)} = \frac{(1+i)^{n-1}}{(1+i)^n} = \frac{1}{1+i}$.

Section 1.4: Present value and Discount

- Assume that a person borrows 1 at an effective rate of discount d.
- The balance $(B) = 1$
- The principal (P): $1 - d$
- The amount of interest (discount): $I = \frac{B-P}{B} = \frac{1-(1-d)}{1} = d$

From definition of the effective rate of interest:

$$i = \frac{I}{P} = \frac{d}{1-d} \Rightarrow i - di = d \Rightarrow d + di = i \Rightarrow d(1+i) = i \Rightarrow d = \frac{i}{1+i}$$

Also,

$$d = \frac{i}{1+i} = i\,\frac{1}{1+i} \Rightarrow d = i\,\nu \quad \text{and} \quad d = \frac{i}{1+i} = \frac{1+i-1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} \Rightarrow d = 1 - \nu$$

Under compound interest,

$$d = i\nu, \quad \nu = \frac{1}{1+i}, \quad d = 1-\nu, \quad i = \frac{d}{1-d}, \quad d = \frac{i}{i+1} \quad \text{and} \quad (1-d)(1+i) = 1$$

Example: If $i = 7\%$, what are d and ν?

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Example: If $i = 7\%$, what are d and ν?

Solution: We know that $d = \frac{i}{1+i} = \frac{0.07}{1+0.07} = 6.5421\%$ and $\nu = \frac{1}{1+i} = \frac{1}{1+0.07} = 0.934579$.

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Example: If $\nu = 0.95$, what are d and i?

Solution: We know that $d = 1 - \nu = 1 - 0.95 = 0.05$ and $\nu = \frac{1}{1+i}$, so $i = \frac{1}{\nu} - 1 = \frac{1-\nu}{\nu} = \frac{1-0.95}{0.95} = 5.2632\%$.

Section 1.4: Present value and Discount

Recall,

■ Under simple interest:

■ $k.a(t) = k(1+it)$: the accumulated value of k at the end of t periods.

■ $k.a^{-1}(t) = \frac{k}{1+it}$: the present value (or discounted value) of k to be paid at the end of t periods. In terms of the amount of discount (d), we have

$$k.a^{-1}(t) = k(1-dt) \text{ for } 0 \leq t \leq \frac{1}{d} .$$

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$$k.a^{-1}(t) = k.\nu^t = k(1-d)^t \text{ for } t \geq 0 .$$

Example: Find the amount which must be invested at a rate of simple discount of 9 % per annum in order to accumulate \$ 1000 at the end of three years.

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■ Under compound interest:

■ $k.a(t) = (1 + i)^t$: the accumulated value of k at the end of t periods.

■ $k.a^{-1}(t) = \frac{k}{(1+i)^t}$: the present value (or discounted value) of k to be paid at the end of t periods. In terms of the amount of discount (d), we have

$$k.a^{-1}(t) = k.\nu^t = k(1 - d)^t \text{ for } t \geq 0 .$$

Example: Find the amount which must be invested at a rate of simple discount of 9 % per annum in order to accumulate \$ 1000 at the end of three years.

Solution: $k\ a^{-1}(t) = k\ (1 - dt) = 1000\ (1 - 0.09(3)) = \$\ 730$.

Section 1.4: Present value and Discount

Recall,

- Under simple interest:

- $k.a(t) = k(1 + it)$: the accumulated value of k at the end of t periods.

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Section 1.5: Nominal rates of interest and Discount

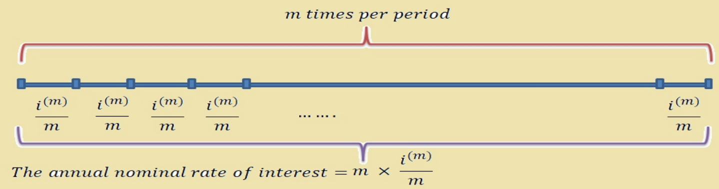
Nominal interest or discount rate: Interest is paid more frequently than once during the measurement period.

Notes:

In some cases, payments are made for a period less than a year (e.g., monthly, quartly, or semi–annually) and in this case, the period interst rate is stated as **a nominal annual rate**= the interest rate per period multiplied by the number of periods per year.

Various terms are used in practice to indicate nominal rates of interest or discount: payable, convertible and compounded.

A nominal annual interest rate is equal to the effective interest rate per period multiplied by the number of periods per year and it takes the symbol $i^{(m)}$, where $m > 1$ is a positive integer. For each m th of a period, the interest rate is $\frac{i^{(m)}}{m}$.



For example: a nominal rate of interest 8% convertible quarterly does not mean 8% per quarter but rather an interest rate of $\frac{8\%}{4} = 2\ \%$ per quarter. If you are to earn 2% compounded quarterly, then $2\% \times 4 = 8\%$ refers to a annual nominal rate converted quarterly. So, an 8% is nominal annual rate convertible quarterly and 2% is quarterly effective rate.

Summary:

1. The interest rate per period: $\frac{i^{(m)}}{m}$.
2. The annual nominal rate: Number of periods per year $\times$ (Rate/period) $= m \times \frac{i^{(m)}}{m}$.
3. The annual effective rate: $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow i = (1 + \frac{i^{(m)}}{m})^m - 1$

Example: Suppose you are earning 2% interest compounded monthly. Find the following:
(1) The annual nominal rate?
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Solution:
(1) The annual nominal rate ($i^{(m)}$): (Rate/period) $\times$ Number of periods per year$= 2\% \times 12 = 0.24 = 24\%$
(2) The annual effective rate (i): $i = (1 + \frac{i^{(m)}}{m})^m - 1 \Rightarrow i = (1 + 2\%)^{12} - 1 = 0.268242 = 26.8242\%$

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Example: John takes a loan of \$ 569. Interest in the loan is charged using compound interest. One month after a loan is taken the balance in this loan is \$ 581.

(1) Find the monthly effective interest rate, which Paul is charged in his loan.

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$$\frac{i^{(12)}}{12} = \frac{581 - 569}{569} = 2.108963093\%$$

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$$i^{(12)} = (12)(0.02108963093) = 25.30755712\%$$

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(3) Find the annual effective rate: $i = (1 + 2.108963093\%)^{12} - 1 = 0.28459549 = 28.459549\%$

Example: If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

Section 1.5: Nominal rates of interest and Discount

Example: If $i^{(4)} = 5\%$ find the equivalent effective annual rate of interest.

Solution: We solve $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + i = (1 + \frac{0.05}{4})^4 \Rightarrow i = (1 + \frac{0.05}{4})^4 - 1 \Rightarrow i = 5.0945\%$.

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$1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + 0.05 = (1 + \frac{i^{(4)}}{4})^4 \Rightarrow 1 + \frac{i^{(4)}}{4} = (1 + 0.05)^{\frac{1}{4}} \Rightarrow i^{(4)} = 4((1 + 0.05)^{\frac{1}{4}} - 1) = 4.9089\%$.

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- Suppose that an account follows compound interest with an annual nominal rate of interest compounded m times a year of $i^{(m)}$, then

- \$ 1 at time zero accrues to $(1+\frac{i^{(m)}}{m})$ at time $\frac{1}{m}$ years.
  - $\frac{i^{(m)}}{m}$: The ($\frac{1}{m}$-year) m–thly effective interest rate.
  - $(1+\frac{i^{(m)}}{m})$: the m-year interest factor.

- \$ 1 at time zero grows to $(1+\frac{i^{(m)}}{m})^m$ in one year.

- \$ 1 at time zero grows to $(1+\frac{i^{(m)}}{m})^{mt}$ in t years.

- The accumulation function is $a(t) = (1+\frac{i^{(m)}}{m})^{mt}$

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Example: John takes a loan of 8,000 at a nominal annual rate of interest of 10% per year convertible quarterly. How much does he owe after 30 months?

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Example: John takes a loan of 8,000 at a nominal annual rate of interest of 10% per year convertible quarterly. How much does he owe after 30 months?

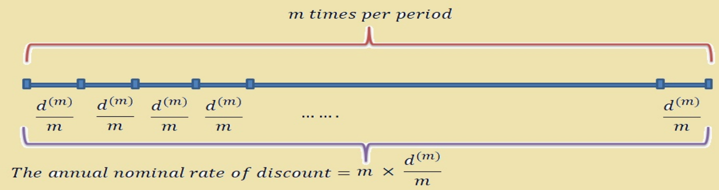
Solution: $ka(t) = k(1 + \frac{i^{(m)}}{m})^{mt} = 8000(1 + \frac{0.10}{4})^{4 \times \frac{30}{12}} = 10240.68$ Note: $\frac{30}{12} = 2.5$ i.e., two years and half.

Section 1.5: Nominal rates of interest and Discount

■ **A nominal annual discount rate** is equal to the effective discount rate per period multiplied by the number of periods per year and it takes the symbol $d^{(m)}$, where $m > 1$ is a positive integer. For each m th of a period, the discount rate is $\frac{d^{(m)}}{m}$.

■ **Notes:**

- For each m th of a period, the discount rate is $\frac{d^{(m)}}{m}$.



- The effective rate of discount (d):

$$1 - d = (1 - \frac{d^{(m)}}{m})^m$$

Remember: $i = \frac{d}{1-d} \Rightarrow 1 + i = 1 + \frac{d}{1-d} = \frac{1-d+d}{1-d} = \frac{1}{1-d} \Rightarrow 1 + i = (1-d)^{-1}$

From this, we have

$$1 + i = (1 + \frac{i^{(m)}}{m})^m = (1-d)^{-1} = (1 - \frac{d^{(m)}}{m})^{-m}$$

Section 1.5: Nominal rates of interest and Discount

- Thus the accumulation function for compound interest under a nominal rate of discount $d^{(m)}$ convertible m times in t years is

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Example: If $d^{(4)} = 5\%$ find i.

Solution: We solve $1 + i = (1 - \frac{d^{(4)}}{4})^{-4}$ to get $i = 5.1602\%$.

$$1 + i = (1 - \frac{d^{(4)}}{4})^{-4} \Rightarrow 1 + i = (1 - \frac{0.05}{4})^{-4}$$
$$\Rightarrow i = (1 - \frac{0.05}{4})^{-4} - 1$$
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Example: If $i = 3\%$ find $d^{(2)}$.

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Example: If $i = 3\%$ find $d^{(2)}$.

Solution: We solve for $d^{(2)}$ in $1 + i = (1 - \frac{d^{(2)}}{2})^{-2}$:

$$1 + i = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1 + 0.03 = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1.03 = (1 - \frac{d^{(2)}}{2})^{-2}$$
$$\Rightarrow (1.03)^{-\frac{1}{2}} = (1 - \frac{d^{(2)}}{2}) \Rightarrow \frac{d^{(2)}}{2} = 1 - (1.03)^{-\frac{1}{2}}$$
$$\Rightarrow d^{(2)} = 2(1 - (1.03)^{-\frac{1}{2}})$$
$$\Rightarrow d^{(2)} = 2.9341\%$$

Section 1.6: Continuous Interest and Force of Interest

Force of interest. The force of interest is the measure of interest at individual moments of time.

The force of interest at time t, denoted by δ_t, is defined as

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt} \ln(A(t)) .$$

To find the force of interest, we may use the accumulation function

$$\begin{aligned} \frac{d}{dt} \ln(A(t)) = \frac{d}{dt} \ln(A(0)a(t)) &= \frac{d}{dt}(\ln(A(0)) + \frac{d}{dt} \ln(a(t)) \\ &= \frac{d}{dt} \ln(a(t)) . \end{aligned}$$

- $\ln(ab) = \ln(a) + \ln(b)$
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Solution: The force of interest is

$$\delta_t = \frac{d}{dt} \ln(A(t)) = \frac{d}{dt} \ln\left(25(1 + \frac{t}{4})^3\right) = \frac{d}{dt}\left(\ln 25 + 3\ln(\frac{4+t}{4})\right) = \frac{d}{dt}\left(3\ln(4+t) - 3\ln(4)\right) = \frac{3}{4+t}$$

- $\ln a^r = r \ \ln a$
- $ln(\frac{a}{b}) = \ln(a) - \ln(b)$
- $\frac{d}{dt} \ln 25 = 0$

Since $\delta_t = \frac{1}{2}$, then

$$\frac{3}{4+t} = \frac{1}{2} \Rightarrow 4 + t = 6 \Rightarrow t = 2$$

Section 1.6: Continuous Interest and Force of Interest

■ **The force of interest under simple and compound interest.**

■ **Under simple interest:** $a(t) = 1 + it$, we have

$$\delta_t = \frac{d}{dt} \ln a(t) = \frac{d}{dt} \ln(1 + it) = \frac{i}{1 + it} \; .$$

Note: The force of interest is decreasing with t.

■ **Under compound interest:** $a(t) = (1 + i)^t$, we have

$$\delta_t = \frac{d}{dt} \ln a(t) = \frac{d}{dt} \ln(1 + i)^t = \frac{d}{dt}\Big(t \;\ln(1 + i)\Big) = \ln(1 + i) \; .$$

Note: Under compound interest, the force of interest is a constant δ, such that

$$\delta = \ln(1 + i) = -\ln \nu \qquad (\text{Remember} \;\; \nu = \frac{1}{1 + i})$$

The force of interest under simple and compound interest.

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Finding the accumulation function using the force of interest.

Theorem: For each $t \geq 0$, $a(t) = e^{\int_0^t \delta_s \ ds}$.

Proof. We know that $\delta_s = \frac{d}{ds} \ln a(s)$ and $a(0) = 1$.

$$\int_0^t \delta_s \ ds = \int_0^t \frac{d}{ds} \ln a(s) = \ln a(s)\Big|_0^t = \ln a(t) - \ln a(0) = \ln a(t) \qquad \text{Remember: } \ln(1) = 0$$

$$\Rightarrow \ln a(t) = \int_0^t \delta_s \ ds$$

$$\Rightarrow a(t) = e^{\int_0^t \delta_s \ ds} \qquad \text{Remember: } e^{\ln a(t) = a(t)}$$

Example: A bank account credits interest using a force of interest $\delta_t = \frac{3t^2}{t^3+2}$. A deposit of 100 is made in the account at time $t = 0$. Find the amount of interest earned by the account from the end of the 4th year until the end of the 8th year.

Example: A bank account credits interest using a force of interest $\delta_t = \frac{3t^2}{t^3+2}$. A deposit of 100 is made in the account at time $t = 0$. Find the amount of interest earned by the account from the end of the 4th year until the end of the 8th year.

Solution: First, we want to find

$$a(t) = e^{\int_0^t \delta_s \; ds}$$

$$\int_0^t \delta_s = \int_0^t \frac{3s^2}{s^3+2} \; ds = \ln(s^3+2)\Big|_0^t = \ln(t^3+2) - \ln(2) = \ln\left(\frac{t^3+2}{2}\right)$$

Remember: $\ln \frac{a}{b} = \ln a - \ln b$

Using

$$a(t) = e^{\int_0^t \delta_s \; ds}$$

we have

$$a(t) = e^{\ln\left(\frac{t^3+2}{2}\right)} = \frac{t^3+2}{2}$$

The amount of interest earned in the considered period is

$$100(a(8) - a(4)) = 100\left(\frac{8^3+2}{2} - \frac{4^3+2}{2}\right) = 22400$$

Example: A bank account credits interest using a force of interest $\delta_t = \frac{3t^2}{t^3+2}$. A deposit of 100 is made in the account at time $t = 0$. Find the amount of interest earned by the account from the end of the 4th year until the end of the 8th year.

Solution: First, we want to find

$$a(t) = e^{\int_0^t \delta_s \ ds}$$

$$\int_0^t \delta_s = \int_0^t \frac{3s^2}{s^3+2} \ ds = \ln(s^3+2)\Big|_0^t = \ln(t^3+2) - \ln(2) = \ln\left(\frac{t^3+2}{2}\right)$$

Remember: $\ln \frac{a}{b} = \ln a - \ln b$

Using

$$a(t) = e^{\int_0^t \delta_s \ ds}$$

we have

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Example: You deposit 1,000 into an account earning a force of interest of 0.06. How long will it take to triple your money?

Solution: First, $a(t) = k.e^{\int_0^t \delta_s \; ds}$.

$$3000 = 1000e^{\int_0^t 0.06s \; ds}$$

$$\Rightarrow 3 = e^{0.06t} \qquad \text{Note: } \int_0^t 0.06 \; ds = 0.06t\Big|_0^t = 0.06t$$

$$\Rightarrow \ln(3) = 0.06t \; \ln(e) \Rightarrow t = frac\ln(3)0.06 = 18.3102 \text{ years}$$

Section 1.12: Solving for PV, FV, n, and i with Compound Interest

Example: You deposit 1,000 into an account earning an annual effective rate of 5%, but with interest payable only at the end of each year. After how many years will the account balance be at least 2,000?

Section 1.12: Solving for PV, FV, n, and i with Compound Interest

Example: You deposit 1,000 into an account earning an annual effective rate of 5%, but with interest payable only at the end of each year. After how many years will the account balance be at least 2,000?

Solution: We know, $a(t) = k.(1+i)^t$.

$$2000 = 1000(1+5\%)^t$$
$$2 = (1.05)^t \Rightarrow \ln(2) = t\ln(1.05)$$
$$\Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = 14.2067$$

Note that the interest is not earned until the end of the year, you will not have 2,000 after 14.2067 years, but you will have more than 2,000 at the end of the 15th year. The answer is therefore 15 years.

Exercises

Exercise 1: **(Practice Exam 6 Exam FM Page PE6-1: Exercise 2)**

The rate at which an investment grows is greatest under which of the following interest scenarios?
A) $d = 0.056$ B) $\delta = 0.057$ C) $d^{(2)} = 0.058$ D) $i^{(4)} = 0.059$ E) $i = 0.060$

Exercises

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A) $d = 0.056$ B) $\delta = 0.057$ C) $d^{(2)} = 0.058$ D) $i^{(4)} = 0.059$ E) $i = 0.060$

Solution:
A) $d = 0.056$

$$1 + i = \frac{1}{1-d} = \frac{1}{1-0.056} = 1.0593 \Rightarrow i = 5.93\%$$

B) $\delta = 0.057$

$$\delta = \ln(1+i) \Rightarrow 1 + i = e^{\delta} = e^{0.057} = 1.0587 \Rightarrow i = 5.87\%$$

C) $d^{(2)} = 0.058$

$$1 + i = (1 - \frac{d^{(2)}}{2})^{-2} \Rightarrow 1 + i = (1 - \frac{0.058}{2})^{-2} = 1.0606 \Rightarrow i = 6.06\%$$

D) $i^{(4)} = 0.059$

$$1 + i = (1 + \frac{i^{(4)}}{4})^{4} = (1 + \frac{0.059}{4})^{4} = 1.0603 \Rightarrow i = 6.03\%$$

E) $i = 0.060$

$$i = 6\%$$

Answer: C

Exercises

Exercise 2: (Practice Exam 6 Exam FM Page PE6-1: Exercise 11)

The balance in an account 1.5 years from today will be 100 (assuming no deposits or withdrawals during that time). Find the current balance if the account earns interest based on a nominal rate of discount of 5% convertible quarterly.
A) 86.8 B) 96.4 C) 92.7 D) 92.9 E) 92.2

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A) 86.8 B) 96.4 C) 92.7 D) 92.9 E) 92.2

Solution: The present value at time 0: $k.a^{-1}(t)$

$$a(t) = \left(1 - \frac{d^{(4)}}{4}\right)^{-(4)(1.5)}$$

Remember: $a(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$

$$\Rightarrow a(t) = \left(1 - \frac{0.05}{4}\right)^{-6} = 1.07839$$

The present value at time 0: $k.a^{-1}(t) = \frac{100}{1.07839} = 92.73$

Answer: C

Exercises

Chapter 1 Exercises:

(1) 1.15: Basic Review Problems: all except 13.

(2) 1.17: Sample exam questions: 1,2,3,4,6,7,8,9.

Sample SOA exam Questions

PE6: 2, 11

PE7: 10, 15, 20