King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S1-1443H
ACTU 475
Credibility Theory and Loss Distributions

Answer the following questions:

(Note that SND Table is attached in page 2)

Q1: [3+2+4]

Consider the model of the total dollars paid on a medical malpractice policy in one year that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-0.3e^{-0.00001x}, & x\geq 0.\end{cases}$$

a) Determine the survival, density, and hazard rate functions.

b) Construct only the graph of the survival function.

c) Determine the mean excess loss and limited expected value functions.

Q2: [8]

The severities of individual claims have a pareto distribution with parameters $\alpha=3$ and $\theta=5{,}000$. Use the central limit theorem to approximate the probability that the sum of 100 independent claims will exceed 300,000.

Q3: [4+4]

a) Find the moment generating function (mgf) and the probability generating function (pgf) for the Poisson distribution.

b) Demonstrate that the transformed beta family as defined by

$$f(x)=\frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$$

is a parametric distribution family.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+2+4]

a) The survival function is

$$S(x) = -F(x)$$
$$\therefore \; S(x) = 0.3e^{-0.00001x}, \quad x \geq 0$$

The density function is

$$f(x) = F^{'}(x) = -S^{'}(x)$$
$$\therefore \; f(x) \; = 0.000003e^{-0.00001x}, \quad x > 0$$

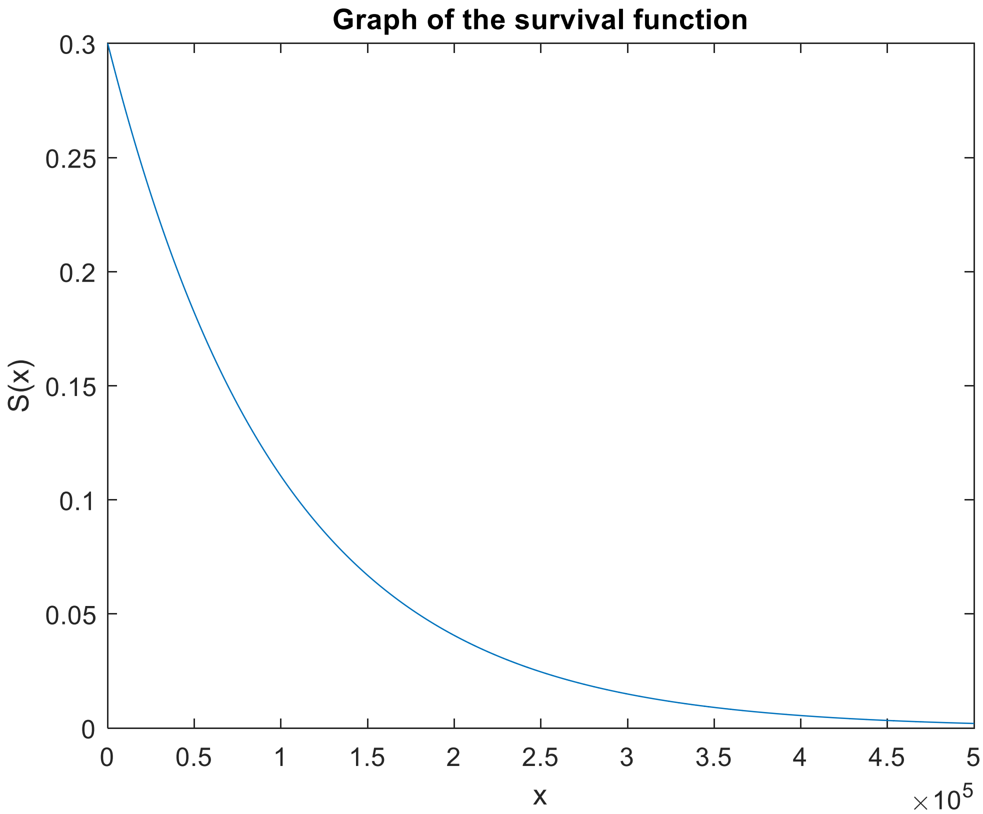
The distribution of this model is mixed, so we can write the probability density function as follows:

$$f(x) = \begin{cases} 0.7, & x = 0, \\ 0.000003e^{-0.00001x}, & x > 0. \end{cases}$$

The hazard rate function

$$h(x) = \frac{f(x)}{S(x)}$$
$$\therefore \; h(x) = 0.00001, \; x > 0$$

b)



c)

The mean excess loss function

$$e(d) = \frac{\int_d^\infty S(x)\ dx}{S(d)}$$

$$= \frac{\int_d^\infty 0.3e^{-0.00001x}\ dx}{0.3e^{-0.00001d}}$$

$$= -100000\frac{[e^{-0.00001x}]_d^\infty}{e^{-0.00001d}}$$

$$\therefore\ e(d) = 100000 \qquad (1)$$

Which is constant function.

To get the limited expected value function $E(X \wedge u)$

First Method

$$E(X \wedge u) = \int_{-\infty}^{u} xf(x)dx + u[1 - F(u)]$$

$$\Rightarrow \; E(X \wedge u) = \int_{0}^{u} x(0.000003)e^{-0.00001x}dx + u(0.3)e^{-0.00001u}$$

$$I = \int_{0}^{u} x(0.000003)e^{-0.00001x}dx$$

$$= 0.000003\left[\frac{xe^{-0.00001x}}{-0.00001}\Bigg|_{0}^{u} - \int_{0}^{u} \frac{e^{-0.00001x}}{-0.00001}dx \right]$$

$$= 0.000003\left[\frac{-ue^{-0.00001u}}{0.00001} - \frac{e^{-0.00001u} - 1}{(0.00001)^2} \right]$$

$$\therefore I \; = -0.3ue^{-0.00001u} - 30000(e^{-0.00001u} - 1) \qquad (2)$$

$$\therefore E(X \wedge u) = 30000\left[1 - e^{-0.00001u}\right]$$

Second Method

By using the following formula

$$E(X \wedge u) = \; E(X) - e(u)S(u)$$

To obtain $E(X)$, let $u \to \infty$ in (2)

$$\therefore \; E(X) = 30000$$

Also, $e(u)$ is determined before in (1), $e(u) = 100000$

$$\therefore \; E(X \wedge u) = 30000 - 100000(0.3e^{-0.00001u})$$
$$= 30000(1 - e^{-0.00001u})$$

Q2: [8]

$$X \sim \text{pareto}(3{,}5000)$$

The Kth moment is given by

$$E(X^k) = \frac{\theta^k \; k!}{(\alpha-1)...(\alpha-k)}$$

$$\therefore \mu = E(X) = \frac{\theta}{\alpha-1}$$

$$= \frac{5000}{2} \; = 2500$$

$$E(X^2) = \frac{\theta^2 \; 2!}{(\alpha-1)(\alpha-2)}$$

$$= \frac{2(5000)^2}{2} = 25.000,000$$

$$Var(X) = E(X^2) - \mu^2$$

$$= 25,000,000 - 2500^2$$

$$= 18.750,000$$

For the sum of random variables

$S_k = X_1 + X_2 + ... + X_k$ where $X_1, X_2, ..., X_k$ are independent random variables

By using central limit theorem, we have

$$E(S_{100}) = 100(2500) = 250,000$$

$$Var(S_{100}) = 100(18750000) = 1,875,000,000$$

$\Rightarrow$ The standard deviation for the sum S_{100} is $\sqrt{1,875,000,000} = 43301.27019$

$$\Pr(S_{100} > 300,000) = 1 - \Phi\left(\frac{300,000 - 250,000}{43301.27019}\right)$$

$$= 1 - \Phi(1.15)$$

$$= 1 - 0.8749 = 0.1251$$

Q3: [4+5]

a) The pgf is

$$P_X(z) = \sum_{x=0}^{\infty} z^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(z\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{z\lambda}$$

$$\therefore P_X(z) = e^{\lambda(z-1)}$$

The mgf is

$$M_X(z) = P_X(e^z) = \exp[\lambda(e^z - 1)]$$

b)

For $X \sim$ Transformed beta $(\alpha, \theta, \gamma, \tau)$ generalized beta

$$f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad (1)$$

$at\ \gamma = \tau = 1$

$$(1) \Rightarrow f(x) = \frac{\Gamma(\alpha+1)(x/\theta)}{\Gamma(\alpha)\Gamma(1)x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha!(x/\theta)}{(\alpha-1)!x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha}{\theta} / \left(\frac{x+\theta}{\theta}\right)^{\alpha+1}$$

$$\therefore f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$$ which is a Pareto Prob. density function (2)

$at\ \tau = 1,\ \gamma = \alpha$

$$(1) \Rightarrow f(x) = \frac{\Gamma(\alpha+1)\alpha(x/\theta)^{\alpha}}{\Gamma(\alpha)\Gamma(1)x[1+(x/\theta)^{\alpha}]^{\alpha+1}}$$

$$\therefore\ f(x) = \frac{\alpha^2(x/\theta)^{\alpha}}{x[1+(x/\theta)^{\alpha}]^{\alpha+1}}$$, which is called paralogistic p.d.f (3)

We can deduce from (1), (2) and (3) that the transformed beta distribution is a parametric distribution family.